TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 270, Number 2, April 1982

RELATIONS BETWEEN H_u^p AND L_u^p WITH POLYNOMIAL WEIGHTS

BY

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ABSTRACT. Relations between L_u^p and H_u^p of the real line are studied in the case when $p > 1$ and $u(x) = |q(x)|^p w(x)$, where $q(x)$ is a polynomial and $w(x)$ satisfies the A_p condition. It turns out that H_u^p and L_u^p can be identified when all the zeros of q are real, and that otherwise H_u^p can be identified with a certain proper subspace of L_u^p. In either case, a complete description of the distributions in H_u^p is given. The questions of boundary values and of the existence of dense subsets of smooth functions are also considered.

1. Introduction. In this paper, we study relations between L_u^p and H_u^p of the real line, where $1 < p < \infty$, $u = u(x)$ is a nonnegative weight function,

$$L_u^p = \left\{ f \colon \left(\int_{\mathbf{R}} |f(x)|^p u(x)\, dx \right)^{1/p} < \infty \right\},$$

and H_u^p is the corresponding Hardy space. To define H_u^p precisely, let $\mathcal{S}$ be the Schwartz space of rapidly decreasing functions, $\mathcal{S}'$ be the space of tempered distributions, and for $l \in \mathcal{S}'$, let Ml denote the nontangential maximal function defined by

$$Ml(x) = (M_{\gamma,\phi} l)(x) = \sup_{(\xi, t) \in \Gamma_\gamma(x)} |\langle l, \phi_t(\xi - \cdot) \rangle|,$$

where $\Gamma_\gamma(x)$ is the cone in $\mathbf{R}_+^2$ of points (ξ, t) with $|x - \xi| < \gamma t$, $\gamma > 0$, $\phi \in \mathcal{S}$, $\phi_t(x) = t^{-1}\phi(x/t)$, $t > 0$, and $\langle l, \psi \rangle$ denotes the action of l on ψ. Then H_u^p is defined to be the collection of $l \in \mathcal{S}'$ such that $M_{\gamma,\phi} l \in L_u^p$ for some $\gamma > 0$ and some ϕ with $\int_{\mathbf{R}} \phi\, dx \neq 0$. If u merely satisfies the doubling condition $\int_{2I} u\, dx \leq c \int_I u\, dx$, where I is an interval, $2I$ is its double and c is a constant independent of I, then the condition that $l \in H_u^p$ is known to be independent of any particular $\gamma > 0$ or $\phi \in \mathcal{S}$ with $\int_{\mathbf{R}} \phi\, dx \neq 0$ (see [9]). We will use the notations

$$\|f\|_{L_u^p} = \|f\|_{p,u} = \left(\int_{\mathbf{R}} |f|^p u\, dx \right)^{1/p}$$

Received by the editors September 30, 1980. The results were presented at the A.M.S. Summer Meeting (August, 1980) in Ann Arbor, Michigan.

1980 *Mathematics Subject Classification.* Primary 30D55; Secondary 42A50, 26C05.

[1] Supported by NSF grants MCS 80-03072 and 77-03980, 80-03098.

0002-9947/81/0000-1048/$10.00

and

$$\|l\|_{H^p_u} = \|M_{\gamma,\phi} l\|_{p,u},$$

for some fixed choice of γ and ϕ, $\int_{\mathbf{R}} \phi\, dx \neq 0$.

If u satisfies the A_p condition

$$\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I u(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq c, \qquad 1 < p < \infty,$$

where c is independent of I, then H^p_u and L^p_u can be identified in the sense that every $f \in L^p_u$ generates a distribution $l_f \in H^p_u$ defined by $\langle l_f, \phi\rangle = \int_{\mathbf{R}} f(x)\phi(x)\,dx$, $\phi \in \mathcal{S}$, and to every $l \in H^p_u$, there corresponds a unique $f \in L^p_u$ such that $l = l_f$. Moreover, in this correspondence, $\|f\|_{L^p_u}$ and $\|l\|_{H^p_u}$ are equivalent. These results are well-known corollaries of the fact (see [**6**]) that the transformation $f \to f^*$, where f^* denotes the Hardy-Littlewood maximal function of f, is bounded on L^p_u, $1 < p < \infty$, if $u \in A_p$.

The weights u considered in this paper for L^p_u and H^p_u will belong to A_r for some $r > p$ but will not belong to A_p. Their specific form is $u(x) = |q(x)|^p w(x)$ where q is a polynomial and $w \in A_p$. Thus, u may have zeros of large orders, and consequently, functions in L^p_u are not generally locally integrable. The motivation for considering such u stems from several places. First, they arise in [**8**] in multiplier questions for L^p_u. Since multiplier results are also derived in [**9**] for H^p_u, we wished to relate L^p_u and H^p_u for this type of u.

As further motivation for studying such u, let us consider a variant of the Hilbert transform. We introduce this variant only for illustration since our proofs do not require any facts about Hilbert transforms. Let $u = |q|^p w$ where q is a polynomial, $1 < p < \infty$, and $w \in A_p$, and define

$$H_q f(x) = \text{p.v.} \int_{\mathbf{R}} f(z)\left[\frac{1}{x-z} - \frac{1}{q(x)}\frac{q(x)-q(z)}{x-z}\right] dz. \tag{1.1}$$

The identity

$$\frac{1}{x-z} - \frac{1}{q(x)}\frac{q(x)-q(z)}{x-z} = \frac{1}{q(x)}\frac{q(z)}{x-z}$$

shows that $H_q f(x) = (fq)^{\sim}(x)/q(x)$, where "$\sim$" denotes the ordinary Hilbert transform. Hence, by the principal result of [**4**], since $w \in A_p$, we have

$$\|H_q f\|_{p,u} = \|(fq)^{\sim}\|_{p,w} \leq c\|fq\|_{p,w} = c\|f\|_{p,u};$$

that is, H_q is bounded on L^p_u.

Next note that if q has degree M, then $(q(x) - q(z))/(x - z)$ is a polynomial in z of degree $M - 1$. Hence, if the first M moments of f exist and equal zero, i.e., if

$$\int_{\mathbf{R}} f(z) z^i\, dz = 0, \qquad i = 0,\ldots,M-1, \tag{1.2}$$

it follows that $H_q f = \tilde{f}$. Condition (1.2) is clearly satisfied for any M if f belongs to the space $\mathcal{S}_{0,0}$ of functions in $\mathcal{S}$ whose Fourier transforms have compact support not containing the origin. Conversely, a result of E. Adams [**1**] states that if u satisfies the doubling condition and the mapping $f \to \tilde{f}$ satisfies $\|\tilde{f}\|_{p,u} \leqslant c\|f\|_{p,u}$ for all $f \in \mathcal{S}_{0,0}$, then u must have the form $u = |q|^p w$ where q is a polynomial and $w \in A_p$. Thus, in a sense, such u are the only weights for which the Hilbert transform is bounded. This can be used to show that if u satisfies A_r for any r and the mapping $f \to Mf$ satisfies $\|Mf\|_{p,u} \leqslant c\|f\|_{p,u}$ for all $f \in \mathcal{S}_{0,0}$, i.e., if

$$\|f\|_{H_u^p} \leqslant c\|f\|_{L_u^p}, \qquad f \in \mathcal{S}_{0,0}, \tag{1.3}$$

then u must have the form above. In fact, the norm boundedness of Mf implies that of the usual (harmonic) Lusin area integral Sf, and since $Sf = S\tilde{f}$, also of $M\tilde{f}$ (see [**3**, **9**]). Since $M\tilde{f}$ pointwise exceeds $|\tilde{f}|$, it follows that the assumptions of Adams' result hold, and therefore that u has the desired form.

The fact that H_q is bounded on L_u^p for such u makes it reasonable to conjecture that L_u^p and H_u^p should coincide. This, however, is not generally the case. The key point to consider turns out to be whether or not all the zeros of q are real. We will systematically use the notation $Q(x)$ to denote a polynomial all of whose zeros are real. The degree of Q will always be denoted N and its distinct zeros will be $\{a_k\}_{k=1}^n$. We normalize Q so that

$$Q(x) = \prod_{k=1}^{n} (x - a_k)^{\mu_k},$$

μ_k being the order (multiplicity) of the zero at a_k. Associated with the partial fraction decomposition of $1/Q$, namely

$$\frac{1}{Q(x)} = \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(x - a_k)^l}, \tag{1.4}$$

will be the distribution

$$\mathfrak{D} = \mathfrak{D}^Q = \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(l-1)!} \delta_{a_k}^{(l-1)}, \tag{1.5}$$

where $\delta_a^{(i)}$ denotes the ith derivative of the delta function at a. $\mathfrak{D}$ is of course supported at the a_k's. If $F(x, y)$ is a function, $\mathfrak{D}_y F$ will denote the action of $\mathfrak{D}$ on F as a function of y. It follows directly from (1.4) and (1.5) that

$$\mathfrak{D}_y^Q(1/(x-y)) = 1/Q(x).$$

If ϕ is a function whose derivatives of order $\mu_k - 1$ exist at a_k, $k = 1, \ldots, n$, define

$$\mathcal{P}_\phi(x) = \mathcal{P}_\phi^Q(x) = Q(x)\mathfrak{D}_y^Q(\phi(y)/(x-y)). \tag{1.6}$$

We will discuss the principal properties of $\mathcal{P}$ in the next two sections. Here, we mention that $\mathcal{P}$ is a polynomial in x whose first $\mu_k - 1$ derivatives at a_k coincide with the corresponding derivatives of ϕ at a_k. Such a polynomial is called an interpolating polynomial; rather than trying to deduce its properties from known facts, we will give direct derivations based on (1.6).

We introduce $\mathcal{P}$ in order to consider modified convolution operators

$$\int_{\mathbf{R}} f(z)\left[\phi(x-z) - \mathcal{P}_{\phi(x-\cdot)}^Q(z)\right] dz.$$

Our motivation again comes from the Hilbert transform. In fact, the term subtracted in (1.1) from the usual kernel $1/(x-z)$ of the Hilbert transform may be written

$$\begin{aligned}\frac{1}{Q(x)}\frac{Q(x)-Q(z)}{x-z} &= \frac{Q(z)}{x-z}\left[\frac{1}{Q(z)}-\frac{1}{Q(x)}\right]\\ &= \frac{Q(z)}{x-z}\left[\mathfrak{D}_y\left(\frac{1}{z-y}\right)-\mathfrak{D}_y\left(\frac{1}{x-y}\right)\right]\\ &= \frac{Q(z)}{x-z}\mathfrak{D}_y\left(\frac{x-z}{(z-y)(x-y)}\right)\\ &= Q(z)\mathfrak{D}_y\left(\frac{1/(x-y)}{z-y}\right) = \mathfrak{P}^Q_{1/(x-\cdot)}(z).\end{aligned}$$

Our main results for the case when q has only real zeros are given in the following two theorems.

THEOREM 1. *Let $1<p<\infty$, Q be a polynomial of degree N with all real zeros, and*

$$f(x,t)=\int_{\mathbf{R}} f(z)\left[\phi_t(x-z)-\mathfrak{P}^Q_{\phi_t(x-\cdot)}(z)\right]dz,$$

where $(1+|x|)^{\alpha+1}|\phi^{(\alpha)}(x)|$ is bounded for $\alpha=0,\dots,N$. If

$$N_\lambda f(x)=\sup_{(\xi,t)\in\mathbf{R}^2_+}\left(\frac{t}{t+|x-\xi|}\right)^\lambda |f(\xi,t)|,$$

then for $\lambda\geq N+1$, $\lambda>1$, there is a constant c independent of f such that $\|N_\lambda f\|_{p,u}\leq c\|f\|_{p,u}$, where $u=|Q|^p w$, $w\in A_p$. In particular, the nontangential maximal function Mf satisfies $\|Mf\|_{p,u}\leq c\|f\|_{p,u}$.

We will also prove a weak-type version of this result when $p=1$.

THEOREM 2. *Let $1<p<\infty$ and $u=|Q|^p w$ where Q is a polynomial with all real zeros and $w\in A_p$. Then H^p_u and L^p_u can be identified in the following sense: there is a unique correspondence between distributions $l\in H^p_u$ and functions $f\in L^p_u$ given by*

$$\langle l,\phi\rangle=\int_{\mathbf{R}} f(z)\left[\phi(z)-\mathfrak{P}^Q_\phi(z)\right]dz,\qquad \phi\in\mathcal{S}.$$

Moreover, in this correspondence $\|l\|_{H^p_u}$ and $\|f\|_{L^p_u}$ are equivalent.

As a corollary of Theorem 2, we see that the $H^p_{|Q|^p w}$ norm of f is equivalent to the H^p_w norm of fQ. We will also show directly in §6 that the function

$$f(x,t)=\langle l_f,\phi_t(x-\cdot)\rangle$$

converges pointwise almost everywhere and in L^p_u norm to f as $t\to 0$.

If q has d complex roots, counting multiplicities, then

$$c_1\leq |q(x)|/\left(1+|x|^2\right)^{d/2}|Q(x)|\leq c_2$$

for positive constants c_1 and c_2, where Q contains all the real zeros of q. Hence, we may assume without loss of generality that

$$u(x)=\left(1+|x|^2\right)^{dp/2}|Q(x)|^p w(x).$$

Again using the Hilbert transform for motivation, note that if $q = q_1 Q$ where q_1 has degree $d \geqslant 1$, then

$$\frac{1}{q(x)}\frac{q(x)-q(z)}{x-z} = \frac{1}{Q(x)}\frac{Q(x)-Q(z)}{x-z} + \frac{1}{q(x)}Q(z)\frac{q_1(x)-q_1(z)}{x-z}.$$

Since $(q_1(x) - q_1(z))/(x - z)$ is a polynomial in z of degree $d - 1$, the second term on the right is the sum of terms of the form $c_i(x)Q(z)z^i$, $i = 0,\ldots,d - 1$. It follows that if

$$\int_{\mathbf{R}} f(z)Q(z)z^i\,dz = 0, \qquad i = 0,\ldots,d-1, \tag{1.7}$$

then

$$H_q f(x) = \text{p.v.}\int_{\mathbf{R}} f(z)\left[\frac{1}{x-z} - \frac{1}{Q(x)}\frac{Q(x)-Q(z)}{x-z}\right]dz = H_Q f(x).$$

Thus, H_Q is bounded on the subset of L_u^p of functions f satisfying (1.7). This subset can be identified with H_u^p as the following result states.

THEOREM 3. *Let* $1 < p < \infty$, *d be a positive integer and* $u = (1 + |x|^2)^{dp/2}|Q|^p w$, *where Q is a polynomial with all real zeros and* $w \in A_p$. *Then H_u^p can be identified with the subspace of L_u^p of f with* $\int_{\mathbf{R}} fQx^i\,dx = 0$ *for* $i = 0,\ldots,d - 1$. *The identification is given by*

$$\langle l, \phi\rangle = \int_{\mathbf{R}} f(z)\left[\phi(z) - \mathcal{P}_\phi^Q(z)\right]dz,$$

$l \in H_u^p$, $f \in L_u^p$, *and* $\|l\|_{H_u^p}$ *is equivalent to* $\|f\|_{L_u^p}$.

As a corollary of Theorem 3, we will see that the H^p norm of f with weight $u = (1 + |x|^2)^{dp/2}|Q|^p w$ is equivalent to the H^p norm of fQ with weight $(1 + |x|^2)^{dp/2}w$. We will also derive the general form for embeddings of L_u^p in H_u^p which are the identity on H_u^p.

In the last section of the paper, we prove several results about the density of $\mathcal{S}_{0,0}$ in H_u^p. While our proofs are direct, we mention that some of the results can be obtained indirectly as corollaries of Theorems 2 and 3 and the density results in **[9]**.

Finally, we list some notation and basic facts. From the definition of A_p, we have that $w \in A_p$, $1 < p < \infty$, if and only if $w^{-1/(p-1)} \in A_{p'}$, $1/p + 1/p' = 1$. Also, from **[4]**, if $w \in A_p$ then

$$\int_{|x|>r}\frac{w(x)}{|x|^p}dx \leqslant \frac{c}{r^p}\int_{|x|<r} w(x)\,dx, \qquad r > 0.$$

In particular, both $w(x)(1 + |x|)^{-p}$ and $w(x)^{-1/(p-1)}(1 + |x|)^{-p'}$ are integrable over $\mathbf{R}$ if $w \in A_p$. If I is an interval and $s > 0$, let sI denote the interval concentric with I whose length is $s|I|$. A weight u is said to satisfy the doubling condition of order $\beta > 0$ if there is a constant c independent of s such that $\int_{sI} u\,dx \leqslant cs^\beta \int_I u\,dx$, $s > 1$. It follows easily that any $w \in A_p$ satisfies this condition with $\beta = p$, and that any u of the form $u = |q|^p w$ where q has degree M and $w \in A_p$ satisfies it with $\beta = (M + 1)p$.

About H_u^p, we will use the fact that $\|N_\lambda l\|_{p,u} \leqslant c\|l\|_{H_u^p}$ if u satisfies the doubling condition of order β and $\lambda > \beta/p$. In particular, if $u \in A_p$, $1 < p < \infty$, the fact that H_u^p and L_u^p can be identified then shows that $\|N_\lambda f\|_{p,u} \leqslant c\|f\|_{p,u}$ if $\lambda > 1$. Finally, we will use the fact that if u satisfies any doubling condition, then a distribution l whose radial maximal function

$$M_0 l(x) = \sup_{t>0} |\langle l, \phi_t(x - \cdot)\rangle|$$

belongs to L_u^p for some $\phi \in \mathcal{S}$ with $\int_{\mathbf{R}} \phi\, dx \neq 0$ also belongs to H_u^p. Proofs of these results may be found in [9].

2. Preliminaries. Let Q and $\mathfrak{D}$ be as in the introduction: that is, let $Q(x) = \prod_{k=1}^n (x - a_k)^{\mu_k}$, a_k real and distinct, $\sum_{k=1}^n \mu_k = N$, and let

$$\mathfrak{D} = \mathfrak{D}^Q = \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(l-1)!} \delta_{a_k}^{(l-1)},$$

where $Q(x)^{-1} = \sum_{k=1}^n \sum_{l=1}^{\mu_k} (A_{k,l}/(x - a_k)^l)$. As noted in the introduction, we have

(2.1) $$\mathfrak{D}_y\big((x-y)^{-1}\big) = (Q(x))^{-1}.$$

LEMMA (2.2).

(i) *If P is a polynomial of degree M, then $\mathfrak{D}_y[(P(x) - P(y))/(x - y)]$ is a polynomial of degree at most $M - 1$.*

(ii) *If $\phi \in C^\infty$, then $\mathfrak{D}_y[(\phi(x) - \phi(y))/(x - y)] \in C^\infty$.*

(iii) *If $\phi \in C^\infty$ and $\mathfrak{B}$ is any distribution of the form $\mathfrak{B} = \sum_{k=1}^n \sum_{l=1}^{\mu_k} B_{k,l} \delta_{a_k}^{(l-1)}$, then $\mathfrak{B}(\phi Q) = 0$.*

PROOF. (i) $(P(x) - P(y))/(x - y)$ is a polynomial in x, y of degree $M - 1$. Applying $\mathfrak{D}_y$ to it produces a polynomial in x of degree at most $M - 1$.

(ii) It is enough to show that $(\phi(x) - \phi(y))/(x - y)$ is infinitely differentiable on $\mathbf{R} \times \mathbf{R}$. This follows easily from the formula

$$\frac{\phi(x) - \phi(y)}{x - y} = \int_0^1 \phi'(y + s(x - y))\, ds.$$

(iii) It is enough to show each $\delta_{a_k}^{(l-1)}(\phi Q) = 0$, $1 \leqslant l \leqslant \mu_k$. By Leibniz's rule,

$$\delta_{a_k}^{(l-1)}(\phi Q) = \sum_{j=0}^{l-1} \binom{l-1}{j} \phi^{(l-1-j)}(a_k) Q^{(j)}(a_k) = 0$$

since $Q^{(j)}(a_k) = 0$ for $0 \leqslant j \leqslant \mu_k - 1$.

LEMMA (2.3). *If $\phi \in C^\infty$, then $Q(x)\mathfrak{D}_y[\phi(y)/(x - y)]$ is a polynomial of degree at most $N - 1$.*

PROOF. It is enough to show that each $Q(x)\delta_{a_k,y}^{(l-1)}[\phi(y)/(x - y)]$, $1 \leqslant l \leqslant \mu_k$, $1 \leqslant k \leqslant n$, is a polynomial of degree at most $N - 1$. But $\delta_{a_k,y}^{(l-1)}[\phi(y)/(x - y)]$ is a linear combination of terms $(x - a_k)^{-i}$, $1 \leqslant i \leqslant l$, and since $Q(x)$ contains the factor $(x - a_k)^{\mu_k}$, the conclusion follows.

We use the notation $\mathcal{P}_\phi(x) = \mathcal{P}_\phi^Q(x) = Q(x)\mathfrak{D}_y^Q[\phi(y)/(x-y)]$ for the polynomial in Lemma (2.3). The next lemma shows that $\mathcal{P}_\phi$ is the interpolating polynomial for ϕ based on Q.

LEMMA (2.4). *Let $\mathfrak{B}$ be any distribution of the form $\mathfrak{B} = \sum_{k=1}^n \sum_{l=1}^{\mu_k} B_{k,l}\delta_{a_k}^{(l-1)}$. If $\phi \in C^\infty$, then $\mathfrak{B}(\phi - \mathcal{P}_\phi) = 0$.*

PROOF. We have

$$\begin{aligned}\phi(x) - \mathcal{P}_\phi(x) &= \phi(x) - Q(x)\mathfrak{D}_y[\phi(y)/(x-y)]\\ &= Q(x)\mathfrak{D}_y[(\phi(x)-\phi(y))/(x-y)] = Q(x)\psi(x)\end{aligned}$$

for some $\psi \in C^\infty$, by (2.1) and Lemma (2.2)(ii). By Lemma (2.2)(iii), $\mathfrak{B}(Q\psi) = 0$, and the proof is complete.

Note that since $\mathcal{P}_\phi$ has degree at most $N-1$, it is the only polynomial of degree at most $N-1$ with $\mathfrak{B}(\phi - \mathcal{P}_\phi) = 0$ for all $\mathfrak{B}$ as above.

We now list properties of $\mathcal{P}$ and $\mathfrak{D}$ which will be useful.

LEMMA (2.5).

(i) *If P is a polynomial of degree at most $N-1$, then $\mathfrak{D}_y[P(y)/(x-y)] = P(x)/Q(x)$. Equivalently, $\mathcal{P}_P^Q(x) = P(x)$ for such P.*

(ii) $\mathcal{P}_{\phi Q}^Q(x) = 0$ *if $\phi \in C^\infty$.*

PROOF. Part (i) follows immediately from the uniqueness of $\mathcal{P}$ mentioned above. Part (ii) is similar, using Lemma (2.2)(iii).

LEMMA (2.6). *If $\phi \in \mathbb{S}$ and Q has degree N, then for any z*

$$\int_{\mathbf{R}} \left[\phi(x-z) - \mathcal{P}_{\phi(x-\cdot)}^Q(z)\right] x^j\,dx = 0, \qquad j = 0,\dots,N-1.$$

PROOF. The integral may be written

$$Q(z)\mathfrak{D}_y\left\{\int_{\mathbf{R}} [\phi(x-z) - \phi(x-y)]x^j\,dx/(z-y)\right\}.$$

We have $\int_{\mathbf{R}}\phi(x-z)x^j\,dx = \int_{\mathbf{R}}\phi(x)(x-z)^j\,dx = P(z)$, where P is a polynomial in z of degree at most j. The expression above then equals

$$Q(z)\mathfrak{D}_y\{(P(z)-P(y))/(z-y)\} = 0 \quad \text{if } j \leq N-1$$

by Lemma (2.5)(i).

Next, we list some relations between the $\mathfrak{D}$'s and $\mathcal{P}$'s associated with $Q(x)$ and $xQ(x)$.

LEMMA (2.7). (i) *If $\phi \in C^\infty$, then $\mathfrak{D}^{xQ}(\phi) = \mathfrak{D}^Q[(\phi(y) - \phi(0))/y]$.*

(ii) *If $\phi \in C^\infty$, then $\mathcal{P}_\phi^{xQ} = \mathcal{P}_\phi^Q + \mathfrak{D}^{xQ}(\phi)\cdot Q$.*

PROOF. (i) Note that $(\phi(y)-\phi(0))/y = \int_0^1 \phi'(sy)\,ds$, so that $(\phi(y)-\phi(0))/y$ is also smooth and equals $\phi'(0)$ at $y=0$. Let $\tilde{\mathfrak{D}}(\phi) = \mathfrak{D}^Q[(\phi(y)-\phi(0))/y]$. Then both $\tilde{\mathfrak{D}}$ and $\mathfrak{D}^{xQ}$ are linear combinations of derivatives of delta functions. To show that $\tilde{\mathfrak{D}}(\phi) = \mathfrak{D}^{xQ}(\phi)$, it is then enough, using the uniqueness of partial fraction decompositions, to show that $\tilde{\mathfrak{D}}_y[1/(x-y)] = \mathfrak{D}_y^{xQ}[1/(x-y)]$. However, by (2.1), $\mathfrak{D}_y^{xQ}[1/(x-y)] = 1/xQ(x)$, while

$$\tilde{\mathfrak{D}}_y\left(\frac{1}{x-y}\right) = \mathfrak{D}_y^Q\left[\frac{1/(x-y)-1/x}{y}\right] = \mathfrak{D}_y^Q\left[\frac{1}{x(x-y)}\right]$$
$$= \frac{1}{x}\mathfrak{D}_y^Q\left(\frac{1}{x-y}\right) = \frac{1}{xQ(x)}$$

by (2.1) again.

To prove (ii), write $x = (x-y)+y$ to obtain

$$x\mathfrak{D}_y^{xQ}\left(\frac{\phi(y)}{x-y}\right) = \mathfrak{D}_y^{xQ}\left(\frac{x\phi(y)}{x-y}\right) = \mathfrak{D}^{xQ}(\phi) + \mathfrak{D}_y^{xQ}\left(\frac{y\phi(y)}{x-y}\right).$$

By (i), the last term on the right equals $\mathfrak{D}_y^Q[\phi(y)/(x-y)]$, and (ii) follows immediately by multiplying the resulting identity by $Q(x)$.

By repeated application of formula (ii), we obtain for any positive integer d

$$\mathfrak{P}_\phi^{x^dQ}(z) = \mathfrak{P}_\phi^Q(z) + \sum_{j=0}^{d-1} \mathfrak{D}^{x^{j+1}Q}(\phi)\cdot z^jQ(z). \tag{2.8}$$

Finally, we list some relations between $\mathfrak{P}$ and ordinary Taylor polynomials.

LEMMA (2.9). *With the usual notation, the following formulas hold*:

(i)
$$\mathfrak{P}_\phi^{(x-a)^l}(z) = \sum_{i=0}^{l-1}\frac{\phi^{(i)}(a)}{i!}(z-a)^i;$$

(ii)
$$\mathfrak{P}_\phi^Q(z) = Q(z)\sum_{k=1}^{n}\sum_{l=1}^{\mu_k}\frac{A_{k,l}}{(z-a_k)^l}\mathfrak{P}_\phi^{(x-a_k)^l}(z);$$

(iii)
$$\mathfrak{P}_\phi^Q(z) = P(z) + Q(z)\sum_{k=1}^{n}\sum_{l=1}^{\mu_k}\frac{A_{k,l}}{(z-a_k)^l}\mathfrak{P}_{\phi-P}^{(x-a_k)^l}(z),$$

where in (iii), $P(z)$ *is any polynomial of degree at most* $N-1$.

PROOF. By definition,

$$\mathfrak{P}_\phi^Q(z) = Q(z)\mathfrak{D}_y^Q\left(\frac{\phi(y)}{z-y}\right) = Q(z)\sum_{k=1}^{n}\sum_{l=1}^{\mu_k}\frac{A_{k,l}}{(l-1)!}\delta_{a_k,y}^{(l-1)}\left(\frac{\phi(y)}{z-y}\right).$$

Hence,

$$\mathfrak{P}_\phi^{(x-a)^l}(z) = (z-a)^l\frac{1}{(l-1)!}\delta_{a,y}^{(l-1)}\left(\frac{\phi(y)}{z-y}\right),$$

and (i) follows by direct computation with Leibniz's rule. Part (ii) follows from the last two formulas by substituting the second into the first. To prove (iii), write $\phi = P + (\phi - P)$ to obtain $\mathfrak{P}_\phi^Q = \mathfrak{P}_P^Q + \mathfrak{P}_{\phi-P}^Q = P + \mathfrak{P}_{\phi-P}^Q$, since P has degree at most $N-1$. Applying (ii) to the last term on the right gives (iii).

Formula (iii) may be used to compare $\mathcal{P}_\phi^Q$ to any P of degree $\leqslant N-1$. We will choose $P = \mathcal{P}_\phi^{x^N}$, the Taylor polynomial of ϕ of order $N-1$ around the origin. The derivatives of a Taylor polynomial satisfy

$$\left(\mathcal{P}_\phi^{x^N}\right)^{(j)} = \mathcal{P}_{\phi^{(j)}}^{x^{N-j}}, \qquad j = 0,\ldots,N-1,$$

which follows easily for $j = 1$ from (i) of the last lemma, and for $j > 1$ by repeated application of the case $j = 1$. Using this and (i) of the lemma, we have

$$\begin{aligned}\mathcal{P}^{(x-a_k)^l}_{\phi-\mathcal{P}_\phi^{x^N}}(z) &= \sum_{j=0}^{l-1} \frac{1}{j!}\left(\phi - \mathcal{P}_\phi^{x^N}\right)^{(j)}(a_k)(z-a_k)^j \\ &= \sum_{j=0}^{l-1} \frac{1}{j!}\left\{\phi^{(j)}(a_k) - \mathcal{P}_{\phi^{(j)}}^{x^{N-j}}(a_k)\right\}(z-a_k)^j \\ &= \sum_{j=0}^{l-1} \frac{1}{j!}\left\{\phi^{(j)}(a_k) - \sum_{i=0}^{N-j-1}\frac{1}{i!}\phi^{(j+i)}(0)a_k^i\right\}(z-a_k)^j.\end{aligned}$$

Hence, by (iii) of the lemma,

$$\mathcal{P}_\phi^Q(z) = \sum_{j=0}^{N-1}\frac{\phi^{(j)}(0)}{j!}z^j + Q(z)\sum_{k=1}^{n}\sum_{l=1}^{\mu_k}\frac{A_{k,l}}{(z-a_k)^l} \tag{2.10}$$
$$\cdot\sum_{j=0}^{l-1}\frac{1}{j!}\left\{\phi^{(j)}(a_k) - \sum_{i=0}^{N-j-1}\frac{1}{i!}\phi^{(j+i)}(0)a_k^i\right\}(z-a_k)^j.$$

3. Basic estimates. The main results of this section concern estimates for the interpolating polynomial $\mathcal{P}_{\phi_t(x-\cdot)}(z)$ and the remainder $\phi_t(x-z) - \mathcal{P}_{\phi_t(x-\cdot)}(z)$ which are uniform in $t > 0$. These estimates are given in Lemma (3.3). We will obtain tempered tangential (and therefore also nontangential) estimates as corollaries. Throughout this section, Q will denote a polynomial of degree N with only real zeros $\{a_k\}$ and $\mathcal{P}_\phi = \mathcal{P}_\phi^Q$. Moreover, by A_1, we mean the class of weights w with $|I|^{-1}\int_I w\,dx \leqslant c\,\mathrm{ess}_I \inf w$.

The following simple preliminary lemma will be useful later.

Lemma (3.1). *If ϕ has N bounded derivatives and $c_\phi = \max_{j=0,\ldots,N}\|\phi^{(j)}\|_\infty$, then*

$$\left|\phi(x) - \mathcal{P}_\phi(x)\right| \leqslant c_\phi |Q(x)|/(1+|x|).$$

Moreover, if $1 \leqslant p < \infty$, $w \in A_p$ and $u = |Q|^p w$, then $\phi - \mathcal{P}_\phi$ belongs to $L^{p'}_{u^{-1/(p-1)}}$, the dual space of L^p_u, with norm bounded by a constant depending on w times c_ϕ. (When $p = 1$, we interpret this to mean that $(\phi - \mathcal{P}_\phi)/Qw$ is bounded by a multiple of c_ϕ.)

Proof. We have $\phi(x) - \mathcal{P}_\phi(x) = Q(x)\psi(x)$, where

$$\psi(x) = \mathcal{D}_y\left(\frac{\phi(x)-\phi(y)}{x-y}\right). \tag{3.2}$$

As noted in Lemma (2.2)(ii) and its proof, $\psi \in C^\infty$ and

$$\frac{\phi(x)-\phi(y)}{x-y} = \int_0^1 \phi'(sx+(1-s)y)\,ds.$$

Applying $\mathfrak{D}_y$ to both sides and recalling that $\mathfrak{D}_y$ has order $N-1$, we see $|\psi(x)| \leq c_\phi$. Moreover, for large $|x|$, formula (3.2) gives $|\psi(x)| \leq c_\phi |x|^{-1}$. Combining estimates, we obtain the first statement of the lemma.

For the second statement, as noted in the introduction,

$$\int_{\mathbf{R}} \frac{w(x)^{-1/(p-1)}}{(1+|x|)^{p'}}\,dx = c^{p'} < \infty$$

if $w \in A_p$, $1 < p < \infty$. The analogue for $p = 1$ is $w(x)^{-1}/(1+|x|) \leq c < \infty$. These easily imply that $|Q(x)|/(1+|x|) \in L^{p'}_{w^{-1/(p-1)}}$ with norm bounded by c, and the second statement follows from the first.

LEMMA (3.3). *Let z and a_{k_0} satisfy $|z-a_{k_0}| = \min_k |z-a_k|$. Then*

(i) $$\sup_{t>0} \left|\phi_t(x-z) - \mathfrak{P}_{\phi_t(x-\cdot)}(z)\right| \leq c\frac{|Q(z)|}{|Q(x)|}\frac{1}{|x-a_{k_0}|},$$

if $|z-a_{k_0}| < \frac{1}{2}|x-a_{k_0}|$;

(ii) $$\sup_{t>0} \left|\mathfrak{P}_{\phi_t(x-\cdot)}(z)\right| \leq c\frac{|Q(z)|}{|Q(x)|}\frac{1}{|z-a_{k_0}|},$$

if $|z-a_{k_0}| > \frac{1}{2}|x-a_{k_0}|$;

(iii) $$|Q(x)| \leq c|Q(z)|,$$

if $|z-a_{k_0}| > \frac{1}{2}|x-a_{k_0}|$.

In all cases, the constant c is independent of x and z; in fact, in (iii), *c depends only on the degree of Q, and in* (i) *and* (ii) *it depends only on Q and the bounds on* $(1+|x|)^{\alpha+1}|\phi^{(\alpha)}(x)|$, $\alpha = 0,\dots,N$.

PROOF. For (iii), note $|x-z| \leq |x-a_{k_0}| + |a_{k_0}-z| \leq 3|z-a_{k_0}| \leq 3|z-a_k|$ for any k. Hence, $|x-a_k| \leq |x-z| + |z-a_k| \leq 4|z-a_k|$. Thus, each factor of $|Q(x)|$ is bounded by 4 times the corresponding factor of $|Q(z)|$.

It is enough to prove (i) and (ii) in case $a_{k_0} = 0$; that this is so follows by denoting $(\tau_a\phi)(x) = \phi(x-a)$ and observing that

(3.4) $$\mathfrak{P}^{\tau_a Q}_{\tau_a\phi}(z) = \left(\tau_a \mathfrak{P}^Q_\phi\right)(z).$$

Choose c_0 so that $|\phi^{(\alpha)}(x)| \leq c_0(1+|x|)^{-\alpha-1}$ for $\alpha = 0,\dots,N$. Then $|\phi_t^{(\alpha)}(x)| = t^{-\alpha-1}|\phi^{(\alpha)}(x/t)| \leq c_0(t+|x|)^{-\alpha-1} \leq c_0|x|^{-\alpha-1}$ for such α. We will first prove (ii) for x in a bounded set, say $|x| \leq A = 10\{\max|a_k|+1\}$. From the definition of $\mathfrak{D}$,

$$\left|\mathfrak{D}_y\left(\frac{\phi_t(x-y)}{z-y}\right)\right| \leq c\max_k \max_{0\leq i+j<\mu_k} \frac{|\phi_t^{(j)}(x-a_k)|}{|z-a_k|^{i+1}}$$
$$\leq cc_0 \max_k \max_{0\leq i+j<\mu_k} \frac{1}{|z-a_k|^{i+1}|x-a_k|^{j+1}}.$$

Since $a_{k_0} = 0$ is the closest a_k to z, $|z - a_k| \geq |z|$ for all k. For the same reason, there is a constant $c > 0$ depending on Q so that if $k \neq k_0$, $|z - a_k| \geq c > 0$. Moreover, if $k = k_0$, $|z - a_{k_0}| = |z| > |x|/2$ by hypothesis. Hence, the last expression is at most

$$\frac{cc_0}{|z|}\left\{\max_{k\neq k_0}\max_{0\leq j<\mu_k}\frac{1}{|x-a_k|^{j+1}}+\max_{0\leq i+j<\mu_{k_0}}\frac{1}{|x|^{i+j+1}}\right\}.$$

Since $|x| \leq A$, this is bounded by

$$\frac{cc_0}{|z|}\max_k\frac{1}{|x-a_k|^{\mu_k}}\leq\frac{cc_0}{|z|}\frac{1}{|Q(x)|}.$$

Using the definition of $\mathcal{P}$, we then obtain (ii) for $|x| \leq A$.

To prove (ii) for large $|x|$, i.e., $|x| > A$, we apply formula (2.10) to the function $\phi_t(x - \cdot)$, x fixed, to obtain

$$\mathcal{P}_{\phi_t(x-\cdot)}(z)=\sum_{j=0}^{N-1}\frac{(-1)^j}{j!}\phi_t^{(j)}(x)z^j+Q(z)\sum_{k=1}^{n}\sum_{l=1}^{\mu_k}\frac{A_{k,l}}{(z-a_k)^l}$$

(3.5)

$$\cdot\sum_{j=0}^{l-1}\frac{(-1)^j}{j!}\left\{\phi_t^{(j)}(x-a_k)-\sum_{i=0}^{N-j-1}\frac{(-1)^i}{i!}\phi_t^{(j+i)}(x)a_k^i\right\}(z-a_k)^j.$$

Now $|z|$ is also large since $|z| > |x|/2$. The first sum on the right of (3.5) is bounded in absolute value by $\sum_{j=0}^{N-1} c_0 |x|^{-j-1} |z|^j$. However,

$$|x|^{-j-1}|z|^j=\left(\frac{|z|}{|x|}\right)^{j+1}\frac{1}{|z|}\leq c\left(\frac{|z|}{|x|}\right)^{N}\frac{1}{|z|}$$

since $j + 1 \leq N$ and $|z| > |x|/2$. Since $|x|$ is large, $|Q(x)| \sim |x|^N$, that is, $|Q(x)|$ is bounded above and below by positive multiples of $|x|^N$. Also $|z|^N \sim |Q(z)|$, and therefore the first sum on the right of (3.5) is bounded by $cc_0 |Q(z)|/|Q(x)||z|$, as desired.

Consider the second term on the right in (3.5). Since $|x|$ is large, Taylor's theorem shows that each term in curly brackets is bounded by a multiple of $|\phi_t^{(N)}|$ evaluated at a point whose absolute value is comparable to $|x|$. Thus, each term in curly brackets is bounded by $cc_0 |x|^{-N-1}$, and so by $cc_0 |x|^{-N}$. Also, since $|z|$ is large and $j \leq l - 1$, $|z - a_k|^{-l+j} \leq c|z|^{-l+j} \leq c|z|^{-1}$. Combining these facts, we see that for $|x| > A$ and $|z| > |x|/2$ the second term on the right of (3.5) is bounded in absolute value by

$$|Q(z)|\,|z|^{-1}cc_0|x|^{-N}\leq cc_0|Q(z)|/|Q(x)|\,|z|.$$

This completes the proof of (ii).

To prove (i) for $|x| < A$, write

$$\phi_t(x-z)-\mathcal{P}_{\phi_t(x-\cdot)}(z)=Q(z)\mathfrak{D}_y\left(\frac{\phi_t(x-z)-\phi_t(x-y)}{z-y}\right).$$

Split $\mathfrak{D}_y = \mathfrak{D}_y^0 + \mathfrak{D}_y^1$ where $\mathfrak{D}^0$ is supported at $a_{k_0} = 0$ and $\mathfrak{D}^1$ at $\{a_k\} \setminus \{0\}$. Since

$$\frac{\phi_t(x-z) - \phi_t(x-y)}{z-y} = -\int_0^1 \phi_t'(x-y-s(z-y))\,ds,$$

we have

$$\left|\mathfrak{D}_y^0\left(\frac{\phi_t(x-z)-\phi_t(x-y)}{z-y}\right)\right| \leqslant c \max_{0\leqslant\alpha<\mu_{k_0}} \int_0^1 \left|\phi_t^{(\alpha+1)}(x-sz)\right| ds$$

(3.6)

$$\leqslant c \max_{\substack{0\leqslant\alpha<\mu_{k_0}\\ |u|\leqslant|z|}} \left|\phi_t^{(\alpha+1)}(x-u)\right|.$$

By assumption, $|u| \leqslant |z| < |x|/2$, and formula (3.6) is therefore bounded by $cc_0 \max_{0\leqslant\alpha<\mu_{k_0}} |x|^{-\alpha-2}$. For $|x| < A$, this is at most

$$cc_0 |x|^{-\mu_{k_0}-1} \leqslant cc_0 |x|^{-1} |Q(x)|^{-1},$$

as desired. Also,

$$\left|\mathfrak{D}_y^1\left(\frac{\phi_t(x-z)-\phi_t(x-y)}{z-y}\right)\right| \leqslant c\left\{|\phi_t(x-z)| + \max_{\substack{k\\ 0\leqslant\alpha<\mu_k}} \left|\phi_t^{(\alpha)}(x-a_k)\right|\right\}$$

since $|z - a_k| \geqslant c > 0$ for $k \neq k_0$. This is at most

$$cc_0\left\{|x-z|^{-1} + \max_{k;\, 0\leqslant\alpha<\mu_k} |x-a_k|^{-\alpha-1}\right\}.$$

We have $|x-z| \geqslant |x| - |z| > |x|/2$. Hence, since x is bounded, the last estimate is at most $cc_0 \max_k |x-a_k|^{-\mu_k}$, so that

(3.7) $$\left|\mathfrak{D}_y^1\left(\frac{\phi_t(x-z)-\phi_t(x-y)}{z-y}\right)\right| \leqslant cc_0 \frac{1}{|Q(x)|}, \qquad |x| < A.$$

This is bounded by $cc_0 |Q(x)|^{-1} |x|^{-1}$ since $|x| < A$, and (i) follows for $|x| < A$.

To prove (i) for $|x| > A$, we use (3.5). Recall that each term there in curly brackets is bounded by $c|\phi_t^{(N)}(\xi)|$ with $|\xi| \sim |x|$, and so is bounded by $cc_0 |x|^{-N-1}$. Furthermore, note that the term in curly brackets corresponding to $k = k_0$ is zero. For $k \neq k_0$, $|z - a_k| \geqslant c > 0$ and therefore $|z - a_k|^{-l+j}$ is bounded since $j \leqslant l - 1$. Hence, for $|x| > A$, the second term on the right of (3.5) is majorized in absolute value by $cc_0 |Q(z)|\,|x|^{-N-1} \leqslant cc_0 |Q(z)|/|Q(x)|\,|x|$.

To complete the proof of (i) for $|x| > A$, it is then enough by (3.5) to estimate the difference between $\phi_t(x-z)$ and the first term on the right of (3.5). Taylor's theorem shows this difference is in absolute value at most $|z|^N |\phi_t^{(N)}(\xi)|$ for some ξ between x and $x - z$. This is bounded by $cc_0 |z|^N |x|^{-N-1}$ since $|z| < |x|/2$. We have $|x|^{-N} \sim |Q(x)|^{-1}$ since $|x|$ is large. Finally, $|z|^N \leqslant c|Q(z)|$ since if $|z|$ is large, $|z|^N \sim |Q(z)|$, while if $|z|$ is bounded, $|z|^N \leqslant c|z|^{\mu_{k_0}} \leqslant c|Q(z)|$ since the other factors of $|Q(z)|$ are all bounded below by positive constants. This completes the proof of the lemma.

As a corollary, we obtain the following estimates.

LEMMA (3.8). *If z and a_{k_0} satisfy $|z - a_{k_0}| = \min_k |z - a_k|$, then*

(i) $$\sup_{(\xi,t)\in\mathbf{R}_+^2} \left(\frac{t}{t+|\xi - x|}\right)^{N+1} \left|\phi_t(\xi - z) - \mathcal{P}_{\phi_t(\xi-\cdot)}(z)\right| \leq c\frac{|Q(z)|}{|Q(x)|}\frac{1}{|x - a_{k_0}|}$$

if $|z - a_{k_0}| < \frac{1}{2}|x - a_{k_0}|$;

(ii) $$\sup_{(\xi,t)\in\mathbf{R}_+^2} \left(\frac{t}{t+|\xi - x|}\right)^{N+1} \left|\mathcal{P}_{\phi_t(\xi-\cdot)}(z)\right| \leq c\frac{|Q(z)|}{|Q(x)|}\frac{1}{|z - a_{k_0}|}$$

if $|z - a_{k_0}| > \frac{1}{2}|x - a_{k_0}|$. *The constant c depends only on Q and the bounds on* $(1 + |x|)^{\alpha+1}|\phi^{(\alpha)}(x)|$, $\alpha = 0,\ldots,N$.

PROOF. Given ϕ, let $\psi(u) = \phi(u + (\xi - x)/t)$. Note that

$$(1 + |u|)^{\alpha+1}|\psi^{(\alpha)}(u)| = (1 + |u|)^{\alpha+1}|\phi^{(\alpha)}(u + (\xi - x)/t)|,$$

so that the sup over u of the expression on the left is at most $(1 + |\xi - x|/t)^{\alpha+1}$ times a multiple of the sup over u of $(1 + |u|)^{\alpha+1}|\phi^{(\alpha)}(u)|$. Since $\psi_t(x - z) = \phi_t(\xi - z)$, it then follows by applying Lemma (3.3)(i) to ψ that if $|z - a_{k_0}| = \min_k |z - a_k| < \frac{1}{2}|x - a_{k_0}|$,

$$\left|\phi_t(\xi - z) - \mathcal{P}_{\phi_t(\xi-\cdot)}(z)\right| \leq c\left(1 + \frac{|\xi - x|}{t}\right)^{N+1}\frac{|Q(z)|}{|Q(x)|}\frac{1}{|x - a_{k_0}|}.$$

Here c depends only on Q and the bounds on $(1 + |u|)^{\alpha+1}|\phi^{(\alpha)}(u)|$, $\alpha = 0,\ldots,N$. This proves part (i) of the lemma. Part (ii) follows in the same way from Lemma (3.3)(ii).

4. Proof of Theorem 1. Let $1 < p < \infty$, $w \in A_p$, Q be a polynomial of degree N with all real zeros and $u = |Q|^p w$. Let

$$f(x,t) = \int_{\mathbf{R}} f(z)\left[\phi_t(x - z) - \mathcal{P}^Q_{\phi_t(x-\cdot)}(z)\right]dz$$

and

$$N_\lambda f(x) = \sup_{(\xi,t)\in\mathbf{R}_+^2}\left(\frac{t}{t+|\xi - x|}\right)^{\lambda}|f(\xi,t)|.$$

Note that if $f \in L_u^p$, then by Lemma (3.1), the integral defining $f(x,t)$ converges absolutely.

For each zero a_{k_0} of Q, define

$$f_{k_0}(z) = \begin{cases} f(z) & \text{if } |z - a_{k_0}| = \min_k |z - a_k|,\\ 0 & \text{otherwise.}\end{cases}$$

By summation, it is enough to prove the theorem for $f = f_{k_0}$. Thus, the estimates of Lemma (3.3) are valid for any z where $f(z) \neq 0$. We may also assume for simplicity that $a_{k_0} = 0$; that we may do so follows by translating variables, using the identity (3.4), and observing that the condition $w(x) \in A_p$ is independent of a translation of x.

We have

$$|f(\xi,t)| \leqslant \int_{|z|<|x|/2} |f(z)|\,\left|\phi_t(\xi - z) - \mathcal{P}_{\phi_t(\xi-\cdot)}(z)\right| dz$$
$$+ \int_{|z|>|x|/2} |f(z)|\,|\phi_t(\xi - z)|\,dz + \int_{|z|>|x|/2} |f(z)|\,\left|\mathcal{P}_{\phi_t(\xi-\cdot)}(z)\right| dz.$$

Multiply both sides of this inequality by $t^\lambda/(t+|x-\xi|)^\lambda$, and let $A(x)$, $B(x)$ and $C(x)$ denote the suprema over $(\xi,t)\in\mathbf{R}^2_+$ of the resulting three terms on the right, resp. Thus, $N_\lambda f(x) \leqslant A(x) + B(x) + C(x)$, and it is enough to estimate the norms of A, B and C.

By Lemma (3.8)(i), if $\lambda \geqslant N+1$,

$$A(x) \leqslant c\frac{1}{|Q(x)|\,|x|}\int_{|z|<|x|/2} |f(z)|\,|Q(z)|\,dz.$$

Therefore,

$$\|A\|_{p,u} \leqslant c\left\|\int_{|z|<|x|} |f(z)|\,|Q(z)|\,dz\right\|_{p,w(x)|x|^{-p}}.$$

Applying Hardy's inequality in the form

$$\left\|\int_{|z|<|x|} g(z)\,dz\right\|_{p,w(x)|x|^{-p}} \leqslant c\|g\|_{p,w}, \qquad w\in A_p$$

(see, e.g., [**5**]), we obtain $\|A\|_{p,u} \leqslant c\|fQ\|_{p,w} = c\|f\|_{p,u}$ as desired.

By Lemma (3.8)(ii), if $\lambda \geqslant N+1$,

$$C(x) \leqslant c\frac{1}{|Q(x)|}\int_{|z|>|x|/2} |f(z)|\,|Q(z)|\frac{1}{|z|}\,dz,$$

$$\|C\|_{p,u} \leqslant c\left\|\int_{|z|>|x|/2} |f(z)|\,|Q(z)|\frac{1}{|z|}\,dz\right\|_{p,w}.$$

Applying the dual version of Hardy's inequality, that is,

$$\left\|\int_{|z|>|x|/2} g(z)\,dz\right\|_{p,w} \leqslant c\|g\|_{p,w(x)|x|^p}, \qquad w\in A_p$$

(see, e.g., [**5**]), we get the desired estimate

$$\|C\|_{p,u} \leqslant c\|f(x)Q(x)/x\|_{p,w(x)|x|^p} = c\|f\|_{p,u}.$$

By Lemma (3.3)(iii),

$$B(x) \leqslant \frac{c}{|Q(x)|}\sup_{(\xi,t)\in\mathbf{R}^2_+}\left(\frac{t}{t+|\xi-x|}\right)^\lambda\int_{|z|>|x|/2} |f(z)Q(z)|\,|\phi_t(\xi-z)|\,dz.$$

Enlarging the domain of integration on the right to all of $\mathbf{R}$, it follows that the sup on the right is at most the usual tempered tangential maximal function of order λ of $|fQ|$ formed with $|\phi_t|$ as the approximation of the identity. As noted in the introduction, this maximal function is bounded on L^p_w if $\lambda > 1$ and $w\in A_p$. Hence, $\|B\|_{p,u} \leqslant c\|fQ\|_{p,w} = c\|f\|_{p,u}$. This completes the proof of the theorem.

Theorem 1 has a weak-type analogue when $p = 1$. To state it, we will use the notation $m_w(E) = \int_E w(x)\,dx$ for the w-measure of a set E.

THEOREM (4.1). *Let $Q(x)$, $f(x,t)$ and $N_\lambda f(x)$ be as in Theorem* 1, *and let $w \in A_1$. If $\lambda \geqslant N+1$ and $\lambda > 1$, then*

$$m_w\{x\colon N_\lambda f(x)|Q(x)| > s\} \leqslant cs^{-1}\|f\|_{1,|Q|w}, \qquad s > 0,$$

where c is independent of f and s.

PROOF. As in the proof of Theorem 1, it is enough to show that $A(x)$, $B(x)$ and $C(x)$ satisfy the estimate. For A, we have

$$A(x) \leqslant c\frac{1}{|Q(x)|\,|x|}\int_{|z|<|x|/2}|f(z)|\,|Q(z)|dz,$$

so that

$$A(x)|Q(x)| \leqslant c\frac{1}{|x|}\int_{|x-z|<2|x|}|f(z)Q(z)|dz \leqslant c(fQ)^*(x),$$

where "*" denotes the Hardy-Littlewood maximal function. Hence,

$$m_w\{x\colon A(x)|Q(x)| > s\} \leqslant m_w\{x\colon (fQ)^*(x) > s/c\} \leqslant \frac{c}{s}\int_{\mathbf{R}}|fQ|w\,dx,$$

by [6], since $w \in A_1$. We also have

$$C(x) \leqslant \frac{c}{|Q(x)|}\int_{|z|>|x|/2}|f(z)|\,|Q(z)|\frac{1}{|z|}dz.$$

Thus,

$$\begin{aligned} m_w\{x\colon C(x)\,|Q(x)| > s\} &\leqslant \frac{1}{s}\int_{\mathbf{R}}C(x)|Q(x)|w(x)\,dx \\ &\leqslant \frac{c}{s}\int_{\mathbf{R}}|f(z)|\,|Q(z)|\left\{\frac{1}{|z|}\int_{|x|<2|z|}w(x)\,dx\right\}dz. \end{aligned}$$

Now using the estimate

$$\frac{1}{|z|}\int_{|x|<2|z|}w(x)\,dx \leqslant \frac{1}{|z|}\int_{|x-z|<3|z|}w(x)\,dx \leqslant cw^*(z) \leqslant cw(z),$$

since $w \in A_1$, we see the last expression is at most $cs^{-1}\|f\|_{1,|Q|w}$ as desired. Finally, recall that $B(x)\,|Q(x)|$ is majorized by a multiple of the usual tempered tangential maximal function of order λ of $|fQ|$ at x, formed with $|\phi_t|$ as approximation of the identity. Denoting this by $N'_\lambda(fQ)(x)$ and using the known fact (see [9]) that if $\lambda > 1$ and $w \in A_1$, then $m_w\{x\colon N'_\lambda(g)(x) > s\} \leqslant (c/s)\|g\|_{1,w}$, we immediately obtain the desired estimate for B. This completes the proof.

5. Theorem 2. In this section, we prove Theorem 2 and deduce several corollaries. As usual, we let $u = |Q|^p w$ where $1 < p < \infty$, Q has only real zeros and $w \in A_p$.

We will first show if $f \in L_u^p$ and l_f is defined by

$$\langle l_f, \phi\rangle = \int_{\mathbf{R}} f(z)\left[\phi(z) - \mathcal{P}_\phi(z)\right]dz, \qquad \phi \in \mathcal{S},$$

$\mathcal{P}_\phi = \mathcal{P}_\phi^Q$, then l_f is a tempered distribution in H_u^p and

$$\|l_f\|_{H_u^p} \leq c\|f\|_{L_u^p} \tag{5.1}$$

with c independent of f. In fact, by the second statement in Lemma (3.1), it follows immediately that l_f defines a tempered distribution. Moreover, since the H_u^p norm of l_f is equivalent to the L_u^p norm of the nontangential maximal function of

$$\langle l_f, \phi_t(x - \cdot)\rangle = \int_{\mathbf{R}} f(z)\left[\phi_t(x-z) - \mathcal{P}_{\phi_t(x-\cdot)}(z)\right] dz,$$

we obtain (5.1) from Theorem 1.

Now let l be any distribution in H_u^p, and let $l(z, s) = \langle l, \psi_s(z - \cdot)\rangle = (l * \psi_s)(z)$ for a fixed $\psi \in \mathcal{S}$ with $\int_{\mathbf{R}} \psi(z)\, dz = 1$. By hypothesis, the L_u^p norm of $l(z, s)$ as a function of z is bounded in s by $\|l\|_{H_u^p}$. Hence, since $p > 1$, there exist $s_m \to 0$ and $f \in L_u^p$ so that $l(z, s_m)$ converges weakly in L_u^p to f and $\|f\|_{L_u^p} \leq \|l\|_{H_u^p}$. Our goal is to show that $l = l_f$.

By Lemma (3.1), $\phi - \mathcal{P}_\phi$ belongs to the dual of L_u^p, so by weak convergence,

$$\int_{\mathbf{R}} l(z, s_m)\left[\phi(z) - \mathcal{P}_\phi(z)\right] dz \to \langle l_f, \phi\rangle \tag{5.2}$$

as $m \to \infty$. Let us denote

$$\langle l(s), \phi\rangle = \int_{\mathbf{R}} l(z, s)\left[\phi(z) - \mathcal{P}_\phi(z)\right] dz,$$

$$\langle \tilde{l}(s), \phi\rangle = \langle l * \psi_s, \phi\rangle = \int_{\mathbf{R}} l(z, s)\phi(z)\, dz.$$

Since $l(z, s) \in L_u^p$, $l(s)$ defines a distribution in H_u^p by what has already been proved. As we will now show, the same is true for $\tilde{l}(s)$. That $\tilde{l}(s)$ defines a distribution follows from the fact that, for $s > 0$, $l(z, s)$ is a locally bounded function of z which is also in L_u^p: thus, $l(z, s) \in L^p_{(1+|x|)^{Np}w}$, while any $\phi \in \mathcal{S}$ belongs to the dual space $L^{p'}_{(1+|x|)^{-Np'}w^{-1/(p-1)}}$ (see the end of the introduction). To see that $\tilde{l}(s) \in H_u^p$, we will show that its radial maximal function

$$M_0(x) = \sup_{t>0} \left|\langle \tilde{l}(s), \phi_t(x - \cdot)\rangle\right|, \qquad \phi \in \mathcal{S},$$

is pointwise less than a constant times the sum of the tangential maximal functions of l formed from ϕ and from ψ. In fact,

$$\langle \tilde{l}(s), \phi_t(x - \cdot)\rangle = (l * \psi_s * \phi_t)(x) = \int_{\mathbf{R}} l(z, s)\phi_t(x - z)\, dz,$$

so that given $M > 0$, there is a constant c with

$$\begin{aligned}\left|\langle \tilde{l}(s), \phi_t(x - \cdot)\rangle\right| &\leq c\frac{1}{t}\int_{\mathbf{R}} |l(z, s)|\left(1 + \frac{|x - z|}{t}\right)^{-M} dz \\ &\leq c\left[\sup_{(z,s)} \left(1 + \frac{|x-z|}{s}\right)^{-\lambda} |l(z, s)|\right] \\ &\quad \cdot \frac{1}{t}\int_{\mathbf{R}} \left(1 + \frac{|x - z|}{s}\right)^{\lambda}\left(1 + \frac{|x - z|}{t}\right)^{-M} dz.\end{aligned}$$

If $t \leqslant s$ and $\lambda > 0$,

$$\frac{1}{t}\int_{\mathbf{R}}\left(1+\frac{|x-z|}{s}\right)^{\lambda}\left(1+\frac{|x-z|}{t}\right)^{-M} dz \leqslant \frac{1}{t}\int_{\mathbf{R}}\left(1+\frac{|x-z|}{t}\right)^{\lambda-M} dz,$$

which is a constant independent of x and t if $M > \lambda + 1$. This shows that if $M > \lambda + 1$, $\lambda > 0$, then

$$\sup_{t \leqslant s}\left|\left\langle \tilde{l}(s), \phi_t(x - \cdot)\right\rangle\right| \leqslant c \sup_{(z,s)}\left(1+\frac{|x-z|}{s}\right)^{-\lambda}|l(z,s)|.$$

The expression on the right is the tangential maximal function of l formed from ψ. To obtain an estimate for $t > s$, we only need to note that $l * \psi_s * \phi_t = (l * \phi_t) * \psi_s$ and repeat the argument above, obtaining for $M > \lambda + 1$, $\lambda > 0$,

$$\sup_{t > s}\left|\left\langle \tilde{l}(s), \phi_t(x - \cdot)\right\rangle\right| \leqslant c \sup_{(z,t)}\left(1+\frac{|x-z|}{t}\right)^{-\lambda}|(l * \phi_t)(z)|.$$

Combining inequalities proves the desired estimate for $M_0(x)$.

Next, we will show that $\tilde{l}(s) = l(s)$. This will prove that $l = l_f$ since the fact that $\int \psi = 1$ implies $\langle \tilde{l}(s), \phi\rangle \to \langle l, \phi\rangle$ as $s \to 0$, while by (5.2), $\langle l(s_m), \phi\rangle \to \langle l_f, \phi\rangle$. Let $l_1(s) = \tilde{l}(s) - l(s)$, so that $l_1 \in H_u^p$ and

$$\langle l_1(s), \phi\rangle = \int_{\mathbf{R}} l(z,s)\mathcal{P}_\phi(z)\,dz. \tag{5.3}$$

We want to show that $l_1(s) \equiv 0$. Write

$$\mathcal{P}_\phi(z) = Q(z)\mathfrak{D}_y[\phi(y)/(z-y)] = \sum_{k=1}^{n}\sum_{l=1}^{\mu_k} B_{k,l}(z)\delta_{a_k}^{(l-1)}\phi,$$

where the $B_{k,l}(z)$ are polynomials of degree at most $N-1$. Since any such polynomial is in $L^{p'}_{(1+|x|)^{-Np'}w}$, we may integrate term-by-term to get

$$l_1(s) = \sum_{k=1}^{n}\sum_{l=1}^{\mu_k} B_{k,l}\delta_{a_k}^{(l-1)}, \quad \text{where } B_{k,l} = \int_{\mathbf{R}} l(z,s)B_{k,l}(z)\,dz.$$

In particular, l_1 has compact support and also, by Lemma (2.4), $\langle l_1(s), \phi - \mathcal{P}_\phi\rangle = 0$. Therefore, $\langle l_1(s), \phi\rangle = \langle l_1(s), \phi - \mathcal{P}_\phi\rangle + \langle l_1(s), \mathcal{P}_\phi\rangle = 0 + \langle l_1(s), \mathcal{P}_\phi\rangle$, and we immediately obtain $l_1(s) = 0$ by applying the following lemma.

LEMMA (5.4). *Let* $1 < p < \infty$ *and* $u(x) = |Q(x)|^p w(x)$ *where* Q *is a polynomial of degree* N *with all real zeros and* $w \in A_p$. *If* l_1 *is a distribution with compact support such that* $l_1 \in H_u^p$, *then* $\langle l_1, x^k\rangle = 0$ *for* $0 \leqslant k \leqslant N-1$.

PROOF. Assume conversely that $\langle l_1, x^k\rangle = c_k \neq 0$ for some $0 \leqslant k \leqslant N-1$. Let $\rho(x) \in C^\infty$ with $\rho(x) = 1$ for $|x| < 1$, $\rho(x) = 0$ for $|x| > 2$ and $0 \leqslant \rho(x) \leqslant 1$. Set $\phi(x) = x^k\rho(x)$. Then $\phi_t(x) = t^{-1}\phi(x/t) = t^{-k-1}x^k\rho(x/t)$, and for t large we have

$$\begin{aligned}\langle l_1, \phi_t\rangle &= t^{-k-1}\langle l_1, x^k\rho(x/t)\rangle = t^{-k-1}\left[\langle l_1, x^k\rangle - \langle l_1, x^k(1-\rho(x/t))\rangle\right]\\ &= t^{-k-1}c_k,\end{aligned}$$

using the fact that $x^k(1-\rho(x/t)) = 0$ on the support of l_1 when t is large. Setting $\psi(x) = \phi(-x)$, we see that the function $l_1(z,t) = \langle l_1, \psi_t(z-\cdot)\rangle$ satisfies $l_1(0,t) = t^{-k-1}c_k$ for large t. Thus,

$$\sup_{\Gamma_\gamma(x)} |l_1(z,t)| \geq |c_k| \left(\frac{|x|}{\gamma} \right)^{-k-1} \quad \text{for } |x| \text{ large.}$$

This, however, contradicts the fact that $l_1 \in H_u^p$ since for large $|x|$ and $k \leq N-1$, $(|x|^{-k-1})^p u(x) \geq c|x|^{(N-k-1)p} w(x) \geq cw(x)$, and $w(x)$ is not integrable at infinity. This completes the proof of the lemma.

The only part of Theorem 2 which remains to be proved concerns the uniqueness of the correspondence between l and f. However, if $g \in L_u^p$ and

$$\int_{\mathbf{R}} f(z)\left[\phi(z) - \mathcal{P}_\phi(z)\right] dz = \int_{\mathbf{R}} g(z)\left[\phi(z) - \mathcal{P}_\phi(z)\right] dz,$$

for $\phi \in \mathcal{S}$, then for any ϕ which is zero in a small neighborhood of each a_k, $\int_{\mathbf{R}} f(z)\phi(z)\,dz = \int_{\mathbf{R}} g(z)\phi(z)\,dz$. Since f and g are locally integrable away from the a_k's, it follows that $f = g$ a.e. This completes the proof of the theorem.

COROLLARY (5.5). *Let $1 < p < \infty$ and $u = |Q|^p w$ where Q has all real zeros and $w \in A_p$. If $l \in H_u^p$ and $l(x,s) = (l * \psi_s)(x)$, $s > 0$, $\psi \in \mathcal{S}$, then*

$$\int_{\mathbf{R}} l(x,s) x^j\,dx = 0, \qquad j = 0, \ldots, N-1.$$

PROOF. As shown in the proof of Theorem 2 (see (5.3) and use the fact that $l_1(s) = 0$),

$$\int_{\mathbf{R}} l(x,s)\mathcal{P}_\phi(x)\,dx = 0.$$

Picking $\phi(x) = x^j \rho(x)$, where $j = 0, \ldots, N-1$ and ρ is a function in $\mathcal{S}$ which equals 1 on the support of $\mathcal{D}$, we have $\mathcal{P}_\phi(x) = \mathcal{P}_{x^j}(x) = x^j$, and the corollary follows.

We remark that Corollary (5.5) can also be derived from Theorem 2 by applying Fubini's theorem and Lemma (2.6).

COROLLARY (5.6). *Let $1 < p < \infty$, Q be a polynomial with all real zeros and $w \in A_p$. If $f \in L^p_{|Q|^p_w}$, $\phi \in \mathcal{S}$ and $\int \phi = 1$, then*

$$\left\| \sup_{\Gamma_\gamma(x)} \left| \int_{\mathbf{R}} f(z)\left[\phi_t(y-z) - \mathcal{P}^Q_{\phi_t(y-\cdot)}(z)\right] dz \right| \right\|_{L^p_{|Q|^p w}}$$

and

$$\left\| \sup_{\Gamma_\gamma(x)} \left| \int_{\mathbf{R}} f(z)Q(z)\phi_t(y-z)\,dz \right| \right\|_{L^p_w}$$

are equivalent.

PROOF. We have $f \in L^p_{|Q|^p w}$ if and only if $fQ \in L^p_w$. Moreover, $\|f\|_{p,|Q|^p w} = \|fQ\|_{p,w}$. The corollary then follows immediately from Theorem 2 and the fact that $\|fQ\|_{p,w}$ is equivalent to the second norm in the statement.

6. Convergence of $f(x, t)$. In this section, we give a proof based on the estimates in §3 of the pointwise convergence a.e. of $f(x, t)$ as $t \to 0$. We also study norm convergence.

THEOREM (6.1). *Let $1 \leqslant p < \infty$ and $u = |Q|^p w$ where Q is a polynomial of degree N with all real zeros and $w \in A_p$. Let ϕ satisfy*

$$n_\phi = \max_{|\alpha| \leqslant N} \left\| (1 + |z|)^{\alpha+1+\varepsilon} \phi^{(\alpha)}(z) \right\|_\infty < \infty \tag{6.2}$$

for some $\varepsilon > 0$. If $f \in L_u^p$ and

$$f(x, t) = \int_{\mathbf{R}} f(z)\left[\phi_t(x - z) - \mathcal{P}^Q_{\phi_t(x-\cdot)}(z)\right] dz,$$

then

$$\lim_{\substack{t \to 0 \\ |x - x_0| < \gamma t}} f(x, t) = \left(\int_{\mathbf{R}} \phi \, dz\right) f(x_0)$$

at any Lebesgue point x_0 of f where $Q(x_0) \neq 0$.

PROOF. The proof is a modified version of the usual Lebesgue point proof of the convergence of approximations of the identity. As usual, we may assume that $x_0 = 0$ and $\int \phi = 1$.

First, consider the case when ϕ is bounded and has compact support. Then if t is small enough and $|x| < \gamma t$, $\phi_t(x - \cdot)$ is supported away from the zeros of Q. Thus, $\mathcal{P}_{\phi_t(x-\cdot)} \equiv 0$ and

$$|f(x, t) - f(0)| = \left| \int_{\mathbf{R}} [f(z) - f(0)] \phi_t(x - z) \, dz \right|.$$

If $|x| < \gamma t$, there exists a constant c so that $|\phi_t(x - z)| \leqslant ct^{-1}\chi_{(-ct, ct)}(z)$, and therefore

$$|f(x, t) - f(0)| \leqslant c \frac{1}{t} \int_{|z| < ct} |f(z) - f(0)| dz,$$

which tends to zero with t since $x_0 = 0$ is a Lebesgue point of f. This completes the proof for this case.

For the general case, write $\phi(x) = \rho(rx)\phi(x) + (1 - \rho(rx))\phi(x) = \phi_1 + \phi_2$, where ρ is a smooth truncation defined by $\rho(x) = 1$ if $|x| < 1$, $\rho(x) = 0$ if $|x| > 2$, and $\rho \in C^\infty$. For fixed r, ϕ_1 has compact support and the corresponding extension $f_1(x, t)$ converges to $(\int \phi_1 \, dx) f(0)$ as shown above. Furthermore, as $r \to 0$, $\int \phi_1 \, dx \to \int \phi \, dx$ and the constant n_{ϕ_2} defined by (6.2) with $\phi = \phi_2$ tends to zero. Hence, to prove the theorem, it is enough to show there is a constant A depending on f and x_0 $(= 0)$ such that

$$\sup_{|x| < \gamma t;\, t < 1} |f(x, t)| \leqslant A n_\phi.$$

Since $n_{\phi(\cdot + \alpha)} \leqslant c n_\phi$ if $|\alpha| < \gamma$, it is enough to show that $\sup_{t<1} |f(0, t)| \leqslant A n_\phi$.

Combining the estimates in Lemma (3.3), we get

$$\left|\phi_t(-z) - \mathcal{P}_{\phi_t(-\cdot)}(z)\right| \leq c_0|Q(z)|\left[\frac{n_\phi}{1+|z|} + |\phi_t(-z)|\right],$$

where c_0 depends on Q and x_0. We also have

$$|\phi_t(-z)| \leq n_\phi t^{-1}(1+|z|/t)^{-1-\varepsilon}.$$

Therefore,

$$\frac{|f(0,t)|}{n_\phi} \leq c_0\int_{\mathbf{R}}|fQ|(1+|z|)^{-1}\,dz + c_0\int_{\mathbf{R}}|fQ|t^{-1}\left(1+\frac{|z|}{t}\right)^{-1-\varepsilon}dz.$$

By Hölder's inequality, the first integral on the right is at most

$$\|f\|_{p,u}\left\|w(z)^{-1/p}(1+|z|)^{-1}\right\|_{p'} \leq A.$$

Since $t^{-1}(1+|z|/t)^{-1-\varepsilon} \leq c_\delta(1+|z|)^{-1}$ if $|z| > \delta > 0$, the second is at most a multiple of

$$\int_{|z|<\delta}|fQ|t^{-1}\left(1+\frac{|z|}{t}\right)^{-1-\varepsilon}dz + \int_{|z|>\delta}|fQ|(1+|z|)^{-1}\,dz.$$

Of these two integrals, the second was shown above to be bounded by A. The first, since $\varepsilon > 0$, is by well-known facts about approximations of the identity bounded by a constant times the Hardy-Littlewood maximal function of $fQ\chi_{(|z|<\delta)}$ evaluated at 0. This is bounded by

$$c\sup_{s<\delta}\frac{1}{s}\int_{|z|<s}|fQ|\,dz \leq c\sup_{s<\delta}\frac{1}{s}\int_{|z|<s}|f|\,dz \leq A$$

for small δ, and the proof is complete.

THEOREM (6.3). *With the same assumptions and notation as in Theorem* (6.1), $f(x,t)$ *converges in* L^p_u *norm to* $(\int_{\mathbf{R}}\phi\,dz)f(x)$ *as* $t \to 0$.

PROOF. For $1 < p < \infty$, $\sup_{t>0}|f(x,t)| \in L^p_u$ by Theorem 1, and the result follows from Theorem (6.1) by the Lebesgue dominated convergence.

For $p = 1$, the proof has two steps, the first being to show that

$$\|f(x,t)\|_{1,u} \leq c\|f\|_{1,u} \tag{6.4}$$

with c independent of f and $t > 0$, $u = |Q|w$, $w \in A_1$. This is based on estimates very much like those in §3. We may assume that $f(z) = 0$ unless $|z - a_{k_0}| = \min_k|z - a_k|$ and that $a_{k_0} = 0$. Split the integral defining $f(x,t)$ into parts with $|z| > |x|/2$ and $|z| < |x|/2$. For the part with $|z| > |x|/2$,

$$\left|\phi_t(x-z) - \mathcal{P}_{\phi_t(x-\cdot)}(z)\right| \leq |\phi_t(x-z)| + \left|\mathcal{P}_{\phi_t(x-\cdot)}(z)\right|$$
$$\leq c\frac{|Q(z)|}{|Q(x)|}\left\{|\phi_t(x-z)| + \frac{1}{|z|}\right\}$$

by Lemma (3.3)(ii) and (iii). The corresponding part of $\| f(x, t)\|_{1,u}$ is then at most

$$c\int_{\mathbf{R}}\left[\int_{|z|>|x|/2} |f(z)|\,|Q(z)|\left(|\phi_t(x-z)| + \frac{1}{|z|}\right) dz\right] w(x)\,dx.$$

By Fubini's theorem and the fact that $|x-z|< 3|z|$ if $|x|< 2|z|$, this is at most

$$c\int_{\mathbf{R}} |f(z)|\,|Q(z)|\left[\int_{\mathbf{R}} |\phi_t(x-z)| w(x)\,dx + \frac{1}{|z|}\int_{|x-z|<3|z|} w(x)\,dx\right] dz$$

$$\leqslant c\int_{\mathbf{R}} |f(z)|\,|Q(z)| w^*(z)\,dz.$$

Since $w \in A_1$, we have $w^*(z) \leqslant cw(z)$ for a.e. z, and therefore the last integral is at most $\| f\|_{1,u}$.

For the part of $f(x, t)$ with $|z|<|x|/2$, $|x|< A$, we claim

$$\left|\phi_t(x-z) - \mathcal{P}_{\phi_t(x-\cdot)}(z)\right| \leqslant c(|Q(z)|/|Q(x)|)[\psi_t(x-z)+1],$$

where $\psi(x) = (1 + |x|)^{-1-\varepsilon}$. In fact, combining (3.6) and (3.7) we get the bound

$$c|Q(z)|\left[\max_{0\leqslant\alpha<\mu_{k_0}} t^{-\alpha-2}\left(1 + \frac{|x|}{t}\right)^{-\alpha-2-\varepsilon} + \frac{1}{|Q(x)|}\right].$$

However, since $(1 + |x|/t)^{-\alpha-1} \leqslant (|x|/t)^{-\alpha-1}$,

$$t^{-\alpha-2}(1 + |x|/t)^{-\alpha-2-\varepsilon} \leqslant \psi_t(x)\,|x|^{-\alpha-1} \leqslant \psi_t(x)\,|Q(x)|^{-1},$$

since $|Q(x)| \leqslant c|x|^{\mu_{k_0}} \leqslant c|x|^{\alpha+1}$ if $\alpha + 1 \leqslant \mu_{k_0}$ and $|x|< A$. The claim above now follows immediately from the fact that $\psi_t(x)$ and $\psi_t(x-z)$ are comparable for $|z|<|x|/2$. The corresponding part of $\| f(x, t)\|_{1,u}$ is then at most

$$c\int_{|x|<A}\left[\int_{|z|<|x|/2} |f(z)|\,|Q(z)|(\psi_t(x-z)+1)\,dz\right] w(x)\,dx$$

$$\leqslant c\int_{|z|<A/2} |f(z)|\,|Q(z)|\left[\int_{\mathbf{R}} \psi_t(x-z) w(x)\,dx + \int_{|x|<A} w(x)\,dx\right] dz.$$

The two inner integrals are bounded by $cw^*(z)$ for $|z|< A/2$, and the entire expression is at most $\| f\|_{1,u}$ since $w \in A_1$.

Finally, for the part of $f(x, t)$ with $|z|<|x|/2$, $|x|> A$, we claim

$$\left|\phi_t(x-z) - \mathcal{P}_{\phi_t(x-\cdot)}(z)\right| \leqslant c(|Q(z)|/|Q(x)|)\psi_t(x-z),$$

where ψ is as above. In fact, as shown in §3, we have the bound $c|Q(z)|\,|\phi_t^{(N)}(\xi)|$ where $|\xi|\sim|x|$. Thus

$$|\phi_t^{(N)}(\xi)| \leqslant ct^{-N-1}(1 + |x|/t)^{-N-1-\varepsilon}$$

$$\leqslant c\psi_t(x)|Q(x)|^{-1},$$

since $|Q(x)|\sim|x|^N$ for large $|x|$. This implies the claim, and the corresponding part of $\| f(x, t)\|_{1,u}$ can be easily estimated as before. This completes the first step of the proof.

Pick $g \in C^{\infty}$ with compact support away from the zeros of Q and $\|f - g\|_{1,u}$ small. That such g exists follows from the local integrability of u: in fact, we may successively approximate f by bounded functions with compact support away from the zeros of Q, continuous functions with such support, functions that are polynomials on the intervals on which they are supported, and finally functions g of the type desired. To complete the proof, we may assume $\int \phi = 1$ and show that $f(x, t) \to f(x)$ in L^1_u. We have

$$\begin{aligned}\|f(x,t) - f(x)\|_{1,u} \\ &\leq \|f(x,t) - g(x,t)\|_{1,u} + \|g(x,t) - g(x)\|_{1,u} + \|g(x) - f(x)\|_{1,u} \\ &\leq 2\|f(x) - g(x)\|_{1,u} + \|g(x,t) - g(x)\|_{1,u}\end{aligned}$$

by (6.4), and it is enough to show $g(x, t) \to g(x)$ in norm.

We will first show that $g(x, t)$ converges uniformly to $g(x)$ on compact sets away from the zeros of Q. Write

$$g(x,t) = \int_{\mathbf{R}} g(z)\phi_t(x-z)\,dz - \int_{\mathbf{R}} g(z)\mathcal{P}_{\phi_t(x-\cdot)}(z)\,dz.$$

The first term on the right converges uniformly to $g(x)$ on compact sets. In the second term,

$$\left|\mathcal{P}_{\phi_t(x-\cdot)}(z)\right| \leq c \max_{k;\, 0 \leq \alpha < \mu_k} \left|\phi_t^{(\alpha)}(x - a_k)\right|$$

since $|z - a_k| \geq c > 0$ for all k if $g(z) \neq 0$. If x is also bounded away from every a_k,

$$\left|\mathcal{P}_{\phi_t(x-\cdot)}(z)\right| \leq c \max_{0 \leq \alpha < N} t^{-\alpha-1}\left(1 + t^{-1}\right)^{-\alpha-1-\varepsilon} \leq ct^{\varepsilon},$$

and therefore the second term above converges uniformly to zero away from the a_k's as $t \to 0$. It follows that the part of $\|g(x, t) - g(x)\|_{1,u}$ with x in a compact set away from the a_k's tends to zero with t.

The part of $\|g(x, t) - g(x)\|_{1,u}$ with x in a small neighborhood of some a_k is at most

$$\int_{|x-a_k|<\delta} |g(x,t)|\,|Q(x)|w(x)\,dx + \int_{|x-a_k|<\delta} |g(x)|\,|Q(x)|w(x)\,dx.$$

The second term is independent of t and small with δ. To estimate the first term, note that if $g(z) \neq 0$ then z is bounded and away from all the a_k's. Since x is very near a_k,

$$\begin{aligned}\left|\phi_t(x-z) - \mathcal{P}_{\phi_t(x-\cdot)}(z)\right| &\leq c\left[1 + \max_{j;\, 0 \leq \alpha < \mu_j} \left|\phi_t^{(\alpha)}(x - a_j)\right|\right] \\ &\leq c \max_{0 \leq \alpha < \mu_k} |x - a_k|^{-\alpha-1} \leq c|Q(x)|^{-1}.\end{aligned}$$

Hence, $|g(x, t)| \leq c\|g\|_1 |Q(x)|^{-1}$, and the first term above is bounded uniformly in t by $c\int_{|x-a_k|<\delta} w(x)\,dx$, which is small with δ.

Finally, for $|x|$ large and $|z|$ bounded, as noted in the first half of the proof,

$$\left|\phi_t(x-z) - \mathcal{P}_{\phi_t(x-\cdot)}(z)\right| \leq c\psi_t(x)|Q(x)|^{-1}.$$

Hence, $|g(x,t)| \leq c\psi_t(x)|Q(x)|^{-1}$ for large $|x|$. Since $g(x)=0$ for $|x|>M$, the part of $\|g(x,t)-g(x)\|_{1,u}$ with $|x|>M$ is at most

$$c\int_{|x|>M}\psi_t(x)w(x)\,dx \leq ct^\varepsilon\int_{|x|>M}\frac{w(x)}{|x|^{1+\varepsilon}}\,dx.$$

This tends to zero with t, and the proof is complete.

7. Theorem 3. In this section, we prove Theorem 3 and identify the embeddings of L_u^p in H_u^p which are the identity on H_u^p. Let $1<p<\infty$, d be a positive integer and $u=(1+x^2)^{dp/2}|Q|^p w$, where Q has all real zeros and $w\in A_p$.

First observe that $\|fQx^i\|_{L^1}\leq c\|f\|_{p,u}$, $i=0,\dots,d-1$, since by Hölder's inequality,

$$\|fQx^i\|_{L^1} \leq \|f\|_{p,u}\left\|x^i w(x)^{-1/p}\left(1+|x|^2\right)^{-d/2}\right\|_{p'} \tag{7.1}$$
$$\leq \|f\|_{p,u}\left\|w(x)^{-1/p}(1+|x|)^{-1}\right\|_{p'} = c\|f\|_{p,u},$$

$i=0,\dots,d-1$. This shows that the moments $\int_{\mathbf{R}} fQx^i\,dx$, $i=0,\dots,d-1$, are finite if $f\in L_u^p$.

If $f\in L_u^p$, then f belongs to both $L^p_{|Q|^p w}$ and $L^p_{|x^dQ|^p w}$. By Theorem 2, it follows that f corresponds to $l_1\in H^p_{|Q|^p w}$ and $l_2\in H^p_{|x^dQ|^p w}$:

$$\langle l_1,\phi\rangle=\int_{\mathbf{R}} f(z)\left[\phi(z)-\mathcal{P}_\phi^Q(z)\right]dz,$$

$$\langle l_2,\phi\rangle=\int_{\mathbf{R}} f(z)\left[\phi(z)-\mathcal{P}_\phi^{x^dQ}(z)\right]dz,$$

$$\|l_1\|_{H^p_{|Q|^p w}}\leq c\|f\|_{L_u^p},\qquad \|l_2\|_{H^p_{|x^dQ|^p w}}\leq c\|f\|_{L_u^p}.$$

Thus,

$$\langle l_1,\phi\rangle-\langle l_2,\phi\rangle=\int_{\mathbf{R}} f(z)\left[\mathcal{P}_\phi^{x^dQ}(z)-\mathcal{P}_\phi^Q(z)\right]dz$$
$$=\sum_{i=0}^{d-1}\mathfrak{D}^{x^{i+1}Q}(\phi)\int_{\mathbf{R}} f(z)Q(z)z^i\,dz$$

by (2.8). Assuming that the moments of fQ of order $0,\dots,d-1$ vanish gives $l_1=l_2$. Calling this common value l, we see that l belongs to both $H^p_{|Q|^p w}$ and $H^p_{|x^dQ|^p w}$, and therefore to H_u^p with $\|l\|_{H_u^p}\leq c\|f\|_{L_u^p}$.

Conversely, if $l\in H_u^p$, the nontangential maximal function of l belongs to both $L^p_{|Q|^p w}$ and $L^p_{|x^dQ|^p w}$. By Theorem 2, there exist f and g, $\|f\|_{L^p_{|Q|^p w}}\leq c\|l\|_{H_u^p}$, $\|g\|_{L^p_{|x^dQ|^p w}}\leq c\|l\|_{H_u^p}$, such that

$$\langle l,\phi\rangle=\int_{\mathbf{R}} f(z)\left[\phi(z)-\mathcal{P}_\phi^Q(z)\right]dz,\qquad \langle l,\phi\rangle=\int_{\mathbf{R}} g(z)\left[\phi(z)-\mathcal{P}_\phi^{x^dQ}(z)\right]dz.$$

Choosing ϕ to vanish near the zeros of x^dQ, we obtain $\mathcal{P}_\phi^{x^dQ}=\mathcal{P}_\phi^Q=0$ and $\int_{\mathbf{R}} f(z)\phi(z)\,dz=\int_{\mathbf{R}} g(z)\phi(z)\,dz$. Therefore, $f=g$ a.e., $\|f\|_{L_u^p}\leq c\|l\|_{L_u^p}$, and if we take the difference of the two formulas for $\langle l,\phi\rangle$ we obtain

$$\int_{\mathbf{R}} f(z)\left[\mathcal{P}_\phi^{x^dQ}(z)-\mathcal{P}_\phi^Q(z)\right]dz=0,\qquad \phi\in\mathcal{S}.$$

Pick $\rho \in \mathcal{S}$, $\rho(x) = 1$ near the support of $\mathcal{D}^{x^d Q}$, and let $\phi(x) = x^i Q(x)\rho(x)$, $i = 0,\dots,d-1$. Then

$$\mathcal{P}_\phi^{x^d Q}(z) - \mathcal{P}_\phi^{Q}(z) = \mathcal{P}_{x^i Q}^{x^d Q}(z) - \mathcal{P}_{x^i Q}^{Q}(z) = z^i Q(z) - 0 = z^i Q(z)$$

by Lemma (2.5)(i) and (ii). Therefore, $\int_{\mathbf{R}} f(z) z^i Q(z)\,dz = 0$, $i = 0,\dots,d-1$, and the theorem follows.

Theorem 3 is valid for $Q \equiv 1$ if we adopt the convention that the interpolating polynomial $\mathcal{P}_\phi^Q$ is zero in this case. This follows by checking the proof. Note that a function f satisfies $f \in L^p_{(1+x^2)^{dp/2}|Q|^p w}$ with the moments of fQ up to order at least $d-1$ all zero if and only if the function $g = fQ$ satisfies $g \in L^p_{(1+x^2)^{dp/2} w}$ with the moments of g up to order at least $d-1$ all equal to zero. Hence, we obtain the following (cf. Corollary (5.6)).

COROLLARY (7.2). *Let $1 < p < \infty$, d be a positive integer, Q be a polynomial with all real zeros and $w \in A_p$. If $f \in L^p_{(1+|x|^2)^{dp/2}|Q|^p w}$ and the moments of fQ of order $0,\dots,d-1$ all vanish, then*

$$\left\| \sup_{\Gamma_\gamma(x)} \left| \int_{\mathbf{R}} f(z)\left[\phi_t(y-z) - \mathcal{P}^Q_{\phi_t(y-\cdot)}(z)\right] dz \right| \right\|_{L^p_{(1+|x|^2)^{dp/2}|Q|^p w}}$$

and

$$\left\| \sup_{\Gamma_\gamma(x)} \left| \int_{\mathbf{R}} f(z) Q(z) \phi_t(y-z)\,dz \right| \right\|_{L^p_{(1+|x|^2)^{dp/2} w}}$$

are equivalent if $\phi \in \mathcal{S}$, $\int \phi = 1$.

COROLLARY (7.3). *Under the same assumptions as in Theorem 3, there is a continuous linear embedding $f \to f - k_f$ of L^p_u onto H^p_u which is the identity on H^p_u and which satisfies*

$$\int_{\mathbf{R}} k_f Q x^i\,dx = \int_{\mathbf{R}} f Q x^i\,dx, \qquad i = 0,\dots,d-1.$$

Conversely, any continuous linear embedding of L^p_u in H^p_u which is the identity on H^p_u has this form.

PROOF. As noted in (7.1), $\| fQx^i \|_{L^1} \leq c \| f \|_{p,u}$ for $i = 0,\dots,d-1$. Pick $\eta_i \in \mathcal{S}$, $i = 0,\dots,d-1$, with $\int_{\mathbf{R}} \eta_i(x) Q(x) x^j\,dx = \delta_{ij}$, where δ_{ij} is the Kronecker δ. That such η_i exist can be seen as follows. First, we use Lemma 2.6, p. 182 of **[2]** to pick $\nu_i \in C_0^\infty$ with support away from the zeros of Q such that $\int_{\mathbf{R}} \nu_i(x) x^j\,dx = \delta_{ij}$, and then we set $\eta_i = \nu_i / Q$. Let

$$g_f(x) = \sum_{i=0}^{d-1} \eta_i(x) \int_{\mathbf{R}} f(t) Q(t) t^i\,dt.$$

Then $\| g_f \|_{p,u} \leq c \| f \|_{p,u}$ and $\int_{\mathbf{R}} g_f Q x^i\,dx = \int_{\mathbf{R}} f Q x^i\,dx$, $i = 0,\dots,d-1$. Hence, $f - g_f \in L^p_u$ with $\| f - g_f \|_{p,u} \leq c \| f \|_{p,u}$, and the first $d-1$ moments of $(f - g_f)Q$ vanish. Therefore, $f - g_f \in H^p_u$ by Theorem 3, and since $g_f = 0$ if $f \in H^p_u$ by

Theorem 3 again, the mapping $f \to f - g_f$ is an embedding of L_u^p into H_u^p which is the identity H_u^p.

Conversely, let S be any embedding of L_u^p in H_u^p which is the identity on H_u^p. If $f \in L_u^p$, choose g_f as above. Then $f - g_f \in H_u^p$, so $S(f - g_f) = f - g_f$, $Sf = f - (g_f - Sg_f)$. The function $g_f - Sg_f$ belongs to L_u^p with norm bounded by a multiple of the norm of f, and for $i = 0,\ldots,d-1$,

$$\int_{\mathbf{R}} (g_f - Sg_f)Qx^i\,dx = \int_{\mathbf{R}} g_f Q x^i\,dx - \int_{\mathbf{R}} Sg_f Q x^i\,dx$$
$$= \int_{\mathbf{R}} fQx^i\,dx - 0 = \int_{\mathbf{R}} fQx^i\,dx,$$

by Theorem 3 since $Sg_f \in H_u^p$. This completes the proof.

8. Density theorems.

THEOREM (8.1). *Let Q be a polynomial with all real zeros. If $1 \leqslant p < \infty$ and $w(x)(1 + |x|)^{-M}$ is integrable for some M, then the class of $g \in \mathcal{S}$ with $\int_{\mathbf{R}} g(x)x^j\,dx = 0$ for $j = 0,\ldots,N-1$ is dense in L_u^p, $u = |Q|^p w$.*

PROOF. Since $w(x)(1 + |x|)^{-M}$ is integrable, $\mathcal{S} \subset L_u^p$. Since u is locally integrable, $\mathcal{S}$ is dense in L_u^p (see the related statement for $p = 1$ in the proof of Theorem (6.3)). Hence, it is enough to show that functions in $\mathcal{S}$ can be approximated in L_u^p by such g.

Let $P_{k,l}(x)$ be the polynomials of degree $N - 1$ with

$$\mathcal{P}_\phi^Q(x) = Q(x)\mathcal{D}_y\left(\frac{\phi(y)}{x-y}\right) = \sum_{k=1}^{n}\sum_{l=1}^{\mu_k}\frac{\phi^{(l-1)}(a_k)}{(l-1)!}P_{k,l}(x), \qquad \phi \in \mathcal{S}. \tag{8.2}$$

The $P_{k,l}$ of course depend on Q but are independent of ϕ. Moreover, for $k' \neq k$, $P_{k,l}(x)$ has a zero of order $\mu_{k'}$ at $x = a_{k'}$. Let $\eta_i(x)$, $i = 1,\ldots,N$, be C^∞ functions supported in $[-1,1]$ with $\int_{\mathbf{R}} \eta_i(x)x^{j-1}\,dx = \delta_{ij}$, $j = 1,\ldots,N$, where δ_{ij} is the Kronecker δ: that such η_i exist was noted in the proof of Corollary (7.3). Given $f \in \mathcal{S}$, let

$$\begin{aligned} c_{k,l} &= \int_{\mathbf{R}} f(x)P_{k,l}(x)\,dx, \qquad k = 1,\ldots,n,\ l = 1,\ldots,\mu_k,\\ f_r(x) &= f(x) - \sum_{k=1}^{n}\sum_{l=1}^{\mu_k} c_{k,l}\eta_l\left(\frac{x-a_k}{r}\right)r^{-l}, \qquad r > 0. \end{aligned} \tag{8.3}$$

We claim that $f_r \in \mathcal{S}$, $\int_{\mathbf{R}} f_r(x)x^j\,dx = 0$ for $j = 0,\ldots,N-1$, and f_r converges to f in L_u^p as $r \to 0$. That $f_r \in \mathcal{S}$ is obvious. For the norm convergence, it is enough to show that

$$\int_{\mathbf{R}} \left|\eta_l\left(\frac{x-a_k}{r}\right)r^{-l}Q(x)\right|^p w(x)\,dx \to 0$$

as $r \to 0$ for $k = 1,\ldots,n$ and $l = 1,\ldots,\mu_k$. But $\eta_l((x-a_k)/r)$ is supported in $|x - a_k| < r$, where $|Q(x)| \leqslant cr^{\mu_k}$. Hence, the last integral is at most $cr^{(\mu_k - l)p}\int_{|x-a_k|<r} w(x)\,dx$, which tends to zero with r since $\mu_k - l \geqslant 0$.

To show $\int_{\mathbf{R}} f_r(x)x^j\,dx = 0$ for $j = 0,\dots,N-1$, note that $\{P_{k,l}(x)\}_{k,l}$ spans the space of polynomials $P(x)$ of degree at most $N-1$: in fact, by Lemma (2.5)(i) and (8.2),

$$P(x) = \mathcal{P}_P^Q(x) = \sum_{k=1}^{n}\sum_{l=1}^{\mu_k} \frac{P^{(l-1)}(a_k)}{(l-1)!} P_{k,l}(x) \tag{8.4}$$

for such P. Thus, it is enough to show that

$$\int_{\mathbf{R}} f_r(x)P_{k',l'}(x)\,dx = 0 \tag{8.5}$$

for $k' = 1,\dots,n$, $l' = 1,\dots,\mu_{k'}$. We have

$$\begin{aligned}\int_{\mathbf{R}} f_r(x)P_{k',l'}(x)\,dx &= \int_{\mathbf{R}} f(x)P_{k',l'}(x)\,dx - \sum_{k=1}^{n}\sum_{l=1}^{\mu_k} c_{k,l}\int_{\mathbf{R}} P_{k',l'}(x)\eta_l\left(\frac{x-a_k}{r}\right)r^{-l}\,dx\\ &= c_{k',l'} - \sum_{k=1}^{n}\sum_{l=1}^{\mu_k} c_{k,l}\int_{\mathbf{R}} P_{k',l'}(x)\eta_l\left(\frac{x-a_k}{r}\right)r^{-l}\,dx.\end{aligned}$$

We will show that for $k = 1,\dots,n$ and $l = 1,\dots,\mu_k$,

$$\int_{\mathbf{R}} P_{k',l'}(x)\eta_l\left(\frac{x-a_k}{r}\right)r^{-l}\,dx = \delta_{kk'}\delta_{ll'},$$

from which (8.5) follows. Let $\tau_{k,l}(x) = \eta_l((x-a_k)/r)r^{-l}$. Changing variables and using the definition of η_l, we obtain

$$\int_{\mathbf{R}} \tau_{k,l}(x)(x-a_k)^{j-1}\,dx = \delta_{l,j}, \tag{8.6}$$

$k = 1,\dots,n$, $l = 1,\dots,N$, $j = 1,\dots,N$. As noted earlier, if $k' \neq k$, $P_{k',l'}$ has a zero of order μ_k at a_k, i.e.,

$$P_{k',l'}(x) = \sum_{j>\mu_k} b_{k',l',j}(x-a_k)^{j-1}, \qquad k' \neq k.$$

Therefore, if $k' \neq k$ and $l = 1,\dots,\mu_k$, (8.6) implies

$$\int_{\mathbf{R}} P_{k',l'}(x)\tau_{k,l}(x)\,dx = 0. \tag{8.7}$$

If $k' = k$, (8.4) gives

$$(x-a_k)^{l'-1} = \sum_{\substack{k''=1\\k''\neq k}}^{n}\sum_{l''=1}^{\mu_{k''}} \alpha_{k'',l''}P_{k'',l''}(x) + P_{k,l'}(x).$$

Thus,

$$\int_{\mathbf{R}} P_{k,l'}(x)\tau_{k,l}(x)\,dx$$

$$= \int_{\mathbf{R}} (x-a_k)^{l'-1}\tau_{k,l}(x)\,dx - \sum_{\substack{k'',l''\\ k''\neq k}} \alpha_{k'',l''}\int_{\mathbf{R}} P_{k'',l''}(x)\tau_{k,l}(x)\,dx$$

$$= \delta_{ll'} - 0 = \delta_{ll'}$$

by (8.6) and (8.7). This completes the proof of the theorem.

The conclusion of Theorem (8.1) can be obtained in another way under the stronger assumptions that $1 \leqslant p < \infty$ and $w \in A_p$. As noted at the beginning of the proof of (8.1), it is enough to show that functions in $\mathbb{S}$ can be approximated in L_u^p by functions in $\mathbb{S}$ whose moments up to order $N-1$ vanish. Consider the extension of f defined in Theorem 1:

$$f_t(x) = \int_{\mathbf{R}} f(z)\Big[\phi_t(x-z) - \mathcal{P}^Q_{\phi_t(x-\cdot)}(z)\Big]\,dz.$$

If $f, \phi \in \mathbb{S}$, it follows from the definition of $\mathcal{P}$ that $f_t \in \mathbb{S}$. Moreover, by Corollary (5.5) (or Lemma (2.6) when $p=1$), f_t satisfies the required moment condition, and if we choose ϕ with $\int_{\mathbf{R}}\phi = 1$, then by Theorem (6.3), f_t converges to f in norm as $t \to 0$. Thus, f_t satisfies all the requirements.

THEOREM (8.8). *Let Q be a polynomial with all real zeros. If $1 \leqslant p < \infty$ and*

$$\lim_{m\to\infty} \frac{1}{m^p}\int_{-m}^{m} w(x)\,dx = 0, \tag{8.9}$$

then $\mathbb{S}_{0,0}$ is dense in L_u^p, $u = |Q|^p w$.

For the proof we need the following fact.

LEMMA (8.10). *Let $1 \leqslant p < \infty$, i be a nonnegative integer, and u be a nonnegative function with*

$$\lim_{m\to\infty} \frac{1}{m^{(i+2)p}}\int_{-m}^{m} u(x)\,dx = 0.$$

If $f \in \mathbb{S}$ and $\int_{\mathbf{R}} f(x)x^j\,dx = 0, j = 0,\dots,i$, then there is a sequence of functions in $\mathbb{S}_{0,0}$ which converges to f in L_u^p.

This is a special case of Theorem 6.13 of [7].

To prove Theorem (8.8), first note (8.9) implies that for $m \geqslant 0$,

$$\int_{2^m \leqslant |x| \leqslant 2^{m+1}} \frac{w(x)}{|x|^M}\,dx \leqslant c(2^m)^{p-M}.$$

Adding these inequalities for $m \geqslant 0$ and using the local integrability of w shows that $w(x)(1+|x|)^{-M}$ is integrable for $M > p$. To complete the proof, we only need to combine Theorem (8.1) with Lemma (8.10) for $i = N - 1$ and $u = |Q|^p w$, noting that

$$\frac{1}{m^{(N+1)p}} \int_{-m}^{m} |Q(x)|^p w(x)\, dx \leq \frac{c}{m^p} \int_{-m}^{m} w(x)\, dx \to 0$$

as $m \to \infty$ by (8.9).

COROLLARY (8.11). *Let Q be a polynomial with all real zeros, $1 < p < \infty$, $w \in A_p$ and $u = |Q|^p w$. Then $\mathcal{S}_{0,0}$ is dense in L_u^p.*

PROOF. Since $w \in A_p$, it satisfies the doubling condition of order p,

$$\int_{-m}^{m} w(x)\, dx \leq c m^p \int_{-1}^{1} w(x)\, dx, \qquad m > 1.$$

If $p > 1$, the fact that $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$ (see [**6**]) then implies that (8.9) holds, and the corollary follows.

The density theorems above have analogues for L_u^p when u has the form $u = (1 + |x|^2)^{dp/2} |Q|^p w$ where d is a positive integer.

THEOREM (8.12). *Let $1 \leq p < \infty$, d be a positive integer and $u = (1 + |x|^2)^{dp/2} |Q|^p w$, where w is a nonnegative function such that $w(x)(1 + |x|)^{-M}$ is integrable for some M and $w(x)^{-1/p}(1 + |x|)^{-1} \in L^{p'}$. Then the class of g in $\mathcal{S}$ with $\int_{\mathbf{R}} g(x)x^j\, dx = 0$ for $j = 0,\ldots,N + d - 1$ is dense in the subspace of L_u^p of f with $\int_{\mathbf{R}} fQx^i\, dx = 0$ for $i = 0,\ldots,d - 1$.*

We remark that the assumptions on w are true if $w \in A_p$.

PROOF. Recall from (7.1) that $\| fQx^i \|_{L^1} \leq c \| f \|_{L_u^p}$, $i = 0,\ldots,d - 1$.

This shows that the moments $\int_{\mathbf{R}} fQx^i\, dx$, $i = 0,\ldots,d - 1$, are finite if $f \in L_u^p$ and that if $f_n \to f$ in L_u^p, then

$$\int_{\mathbf{R}} f_n Q x^i\, dx \to \int_{\mathbf{R}} fQx^i\, dx, \qquad i = 0,\ldots,d - 1.$$

We first show that if $f \in L_u^p$ and $\int_{\mathbf{R}} fQx^i\, dx = 0$, $i = 0,\ldots,d - 1$, then f can be approximated in L_u^p by $g \in \mathcal{S}$ with $\int_{\mathbf{R}} gQx^i\, dx = 0$, $i = 0,\ldots,d - 1$. In fact, pick $f_n \in \mathcal{S}$ with $f_n \to f$ in L_u^p, and let

$$g_n(x) = f_n(x) - \sum_{i=0}^{d-1} \eta_i(x) \int_{\mathbf{R}} f_n(t) Q(t) t^i\, dt,$$

where $\{\eta_i\}$ is chosen with $\eta_i \in \mathcal{S}$ and $\int_{\mathbf{R}} \eta_i(x)Q(x)x^j\, dx = \delta_{ij}$. (See the proof of Corollary (7.3).) Clearly, $g_n \in \mathcal{S}$ and

$$\int_{\mathbf{R}} g_n Q x^j\, dx = \int_{\mathbf{R}} f_n Q x^j\, dx - \int_{\mathbf{R}} f_n Q t^j\, dt = 0,$$

$j = 0,\ldots,d - 1$. Moreover, the facts that $f_n \to f$ in L_u^p, $\eta_i \in L_u^p$ and $\int_{\mathbf{R}} f_n Q t^i\, dt \to \int_{\mathbf{R}} fQt^i\, dt = 0$, $i = 0,\ldots,d - 1$, show that $g_n \to f$ in L_u^p, as desired.

Thus, it is enough to prove the theorem for $f \in \mathcal{S}$. Let g_r be the function f_r defined in (8.3) but now formed using $x^d Q(x)$ as generating polynomial. Then, as shown in the proof of Theorem (8.1), $g_r \in \mathcal{S}$, the moments of g_r of order $\leq N + d - 1$ vanish, and $g_r \to f$ in $L^p_{|x^d Q|^p w}$ as $r \to 0$. Since $u \leq c[|Q|^p w + |x^d Q|^p w]$, the proof of the theorem will be complete if we show that g_r also converges to f in $L^p_{|Q|^p w}$. Let f_r

denote the function in (8.3) formed with Q as generator and the same $\{\eta_i\}$ as for g_r. Then $f_r \to f$ in $L^p_{|Q|^p w}$, so we will be done if we show that $f_r = g_r$. However, it follows easily from (8.2) and (2.8) that those polynomials $P_{k,l}$ for $x^d Q$ with $k = 1,\ldots,n$, $l = 1,\ldots,\mu_k$, differ from the corresponding $P_{k,l}$ for Q by linear combinations of the $x^j Q$, $j = 0,\ldots,d-1$, while those $P_{k,l}$ for $x^d Q$ with $k > n$ or $l > \mu_k$ are linear combinations of the $x^j Q$. Hence, since $\int_{\mathbf{R}} f x^j Q\,dx = 0$ for $j = 0,\ldots,d-1$, we obtain $f_r = g_r$.

THEOREM (8.13). *Let $1 \leqslant p < \infty$, d be a positive integer and*

$$u = \left(1 + |x|^2\right)^{dp/2} |Q|^p w,$$

where w is a nonnegative function such that $w(x)^{-1/p}(1 + |x|)^{-1}$ is in $L^{p'}$ and (8.9) *holds. Then $\mathcal{S}_{0,0}$ is dense in the subspace of L_u^p of f with $\int_{\mathbf{R}} fQx^i\,dx = 0$ for $i = 0,\ldots,d-1$.*

PROOF. This follows from the previous theorem by using Lemma (8.10), with i there taken to be $N + d - 1$, in the same way that Theorem (8.8) follows from Theorem (8.1).

COROLLARY (8.14). *If $1 < p < \infty$, d is a positive integer and $u = (1 + |x|^2)^{dp/2} |Q|^p w$ with $w \in A_p$, then $\mathcal{S}_{0,0}$ is dense in the subspace of L_u^p of f with $\int_{\mathbf{R}} fQx^i\,dx = 0$ for $i = 0,\ldots,d-1$. In particular, the closure of $\mathcal{S}_{0,0}$ in L_u^p is this subspace.*

PROOF. The first statement follows as in the proof of Corollary (8.11). For the second, note that if f is in the closure of $\mathcal{S}_{0,0}$, then there exist f_n in $\mathcal{S}_{0,0}$ such that $f_n \to f$ in L_u^p, and therefore, $\int_{\mathbf{R}} f_n Q x^i\,dx \to \int_{\mathbf{R}} fQx^i\,dx$, $i = 0,\ldots,d-1$. Since the integrals on the left all vanish, so does the one on the right.

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